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Complex Hesitant Fuzzy Set-Based Multi-Criteria Decision-Making using Hamacher Aggregation Operators

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ABSTRACT

Multi-criteria decision-making (MCDM) has become a critical tool for addressing real-world decision-making problems under uncertainty. Classical extensions of fuzzy sets (FSs), such as hesitant fuzzy sets (HFS) and complex hesitant fuzzy sets (CHFS), provide a flexible framework for modelling expert judgment ambiguity. Nevertheless, none of the aggregation operators (AOs) for Hamacher t-norms and t-conorms have been developed within the CHFS framework, thereby limiting their theoretical richness and practical use. To fill this void, this paper presents Hamacher AOs in the framework of CHFS that are complex hesitant fuzzy (CHF) Hamacher weighted averaging (CHFHWA) operator, CHF Hamacher weighted geometric (CHFHWG) operator, and their generalizations (generalized CHFHWA and generalized CHFHWG). Moreover, a systematic MCDM approach is developed using newly defined operators to address complex decision problems with multiple and hesitant criteria. The practical applicability and efficiency of the proposed approach are illustrated by a real-life inspired numerical example of student scholarship selection. The comparative analysis with existing models shows that the proposed operators not only increase the flexibility of information aggregation but also improve decision-making accuracy and robustness.

1. Introduction

Recently, MCDM has become more visible as a decision-making approach [1, 3]. This comes with its own challenges: the aggregation functions are vital for making accurate choices, but they often prove an obstacle. Different theories of decision making, such as Zadeh's fuzzy set (FS) theory [4] or its higher order augmentations, such as interval-valued fuzzy sets (IVFSs) [5], and other more generalized approaches, such as intuitionistic fuzzy set (IFS) theory [6] and IVIFSs theory [7], may produce different effects. However, the persistence of a single membership grade and a non-membership grade may be seen as a limitation of these theories.

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This problem was successfully solved by Torra and Narukawa [8] after they were introduced to the hesitant fuzzy set (HFS) theory, which allows multiple membership degrees to be assigned to elements. Consequently, this reduced or even eliminated uncertainty regarding expert preferences [8, 9], allowing each expert to rely on his or her own grading in the decision-making process. An essential contribution was also made by Xia and Xu [10], who defined weighted averaging/geometric AOs within the framework of HFS. These operators were used to solve decision-making concerns. In addition, Xia and Xu [11] build on the distance and similarity measures presented in their previous work and also include weighted distance and weighted similarity measures for HFSs. On top of that, Xia *et al.* [12] expanded the range of studies to include techniques aimed at aggregation operators (AOs) for group decision making in the HFS context. Wei [13] also advanced the field further by showing MCDM for HFS in the form of focused aggregation operators [14, 15]. Zhu *et al.* [16] first described the Bonferroni operator, as well as the weighted Bonferroni operator that relates to it. Senapati *et al.* [17] introduced novel Aczel-Alsina operations-based hesitant fuzzy aggregation operators and their applications in cyclone disaster assessment. In succession, Zhang [18] dealt with a range of hesitant fuzzy (HF) power AOs: these included the HF power weighted averaging and geometric operators, the generalized HF power weighted averaging and geometric operators, all of which are suited for dealing with MCGDM dilemmas.

Further, Beliakov *et al.* [19] proposed an Archimedean norm function, and this was later suggested by Xia *et al.* [20]. They used them in their work on IFS and intuitionistic fuzzy weighted operators. Papers by Yu [21] focused on generalized attention, focusing on generalized operators supported by Archimedean norms. Tan *et al.* [22] proposed the weighted quasi-weight and hybrid-weight approaches for using Archimedean norms, but encountered some setbacks. These efforts have been complemented by the work of Wang and Liu [23, 24], who proposed intuitive fuzzy Einstein AOs, and by Zhao and Wei, who developed hybrid Einstein aggregation operators for IFSs in MCDM [25]. However, these operators only offered one membership value and a non-membership degree. To address this issue, Yu [26] has advanced the study of the Einstein norms of HFSs. Hamacher [27] proposed another type of t-norms and t-conorm called Hamacher t-norms and t-conorm. As the generalization of the algebraic product and sum, and Einstein t-conorm and t-norm is more general and more flexible [27].

Tan *et al.* [28] made significant inroads by using the Hamacher t-norm and t-conorm for HFS, making it easier for decision-makers to define the most important attributes in MCDM. The introduction of the Hamacher t-norm and t-conorm for HFS fostered further advancement, including the development of the complex fuzzy set (CFS) by Ramot *et al.* [29], who proposed a two-dimensional polar representation for MCDM. Mahmood *et al.* [30] generalized HFSs and their applications using exponential and non-exponential similarity measures. Ali *et al.* [31] proposed selecting Industry 4.0 technologies using Aczel-Alsina information based on circular bipolar complex fuzzy uncertainty. The polar orientation of CFS provided the membership values narrowly limited to the unit circle of the complex plane, but Tamir *et al.* [32] circumvent this limitation in their work by presenting CFS in a Cartesian system, defined by the fact that the member values are within a square placed in the complex plane. Such a formal construction is said to be more elegant and easier to use in complex data architectural manipulations. Yazdanbakhsh and Dick [33] reviewed the literature on CFS and reasoning, and Garg *et al.* [34] discussed the structural representation of the theory of complex hesitant fuzzy sets (CHFS) in polar coordinates, which enhances the quality and reliability of several criteria decision-making for complex and fuzzy information. After that, Albaity *et al.* [35] devised the theory of CHFS in Cartesian form, in which they replace the concept of the unit circle by the unit square.

1.1. Motivation and contribution

CHFS has become a powerful mathematical tool for managing uncertainty and ambiguity in complex decision-making situations in the fast-moving field of fuzzy set (FS) theory and its applications. Nevertheless, despite major theoretical progress in this field, there remains a serious research gap: no aggregation operators (AOs) grounded in the Hamacher t-norm and t-conorm have been developed for CHFS. This inherent constraint poses a significant challenge to the practical application and theoretical development of CHFS. The lack of Hamacher-based AOs in the CHFS framework poses several serious challenges. First of all, it does not allow for the successful aggregation of information stored in CHFSs in accordance with the established Hamacher operational principles. This limitation is a serious constraint on researchers' and practitioners' capacity to simultaneously exploit the computational and theoretical benefits of both CHFS and the strong mathematical properties of Hamacher t-norms and t-conorms. The Hamacher operations are especially appreciated for their flexibility, mathematical elegance, and better performance in handling different kinds of uncertain information, so their absence in CHFS is a significant theoretical and practical weakness. Recognizing this important gap in the current literature, there is a strong and pressing need to develop advanced AOs within the CHFS framework that are essentially based on Hamacher t-norm and t-conorm principles. This development would not only fill the current gap in theory but also open new opportunities to improve decision-making in complex, uncertain environments.

To address this important research problem systematically and make a valuable contribution to the development of FS theory, this article proposes a comprehensive set of novel operators. In particular, we define and rigorously develop the CHFHWA and CHFHWG operators, and their generalized versions: the generalized CHFHWA (GCHFHWA) operator and the generalized CHFHWG operator. All these operators are designed with care to preserve the mathematical rigor desired in FS theory while also offering the unique benefits of Hamacher operations. It is based on the theoretical framework developed using these newly created AOs, and this article then presents a detailed and strong multi-criteria decision-making (MCDM) methodology. The methodology is a clever application of the recently developed Hamacher-based AOs to solve complex decision-making problems characterized by numerous conflicting criteria and high uncertainty. To demonstrate the practicality and computational efficiency of our proposed approach, we present a comprehensive numerical example that illustrates its step-by-step implementation in a real-life decision-making context.

Moreover, to demonstrate the credibility and excellence of our offered work, we conduct a comprehensive comparative analysis with approaches and methodologies reported in the literature. This detailed comparison not only reveals the unique benefits and improved performance features of our suggested operators but also shows that they can be practically applied to various decision-making situations. The comparative analysis is used to place our contribution in the broader context of fuzzy set theory and multi-criteria decision-making studies, thereby defining its relevance and potential for future development in this area.

1.2. Framework of the manuscript

In Section 2, we revise a few fundamental theories that help us develop new theories. In Section 3, we devise AOs within the CHFS framework, namely CHFHWA, CHFHWG, GCHFHWA, and GCHFHWG, and discuss related results. In this section, we interpret an MADM approach and then discuss a numerical problem using it. In the section, we have a comparison, and in section 6, we have a conclusion.

2. Preliminaries

In this section, we revise a few fundamental concepts related to the work presented here.

Definition 1: [35] A mathematical form of CHFS π on $\overleftarrow{\mathbb{X}}$ is $\pi = \left\{\left(\underline{\eth}, \overline{\mho}_{\pi}\left(\underline{\eth}\right)\right) | \underline{\eth} \epsilon \overleftarrow{\mathbb{X}}\right\}$, where $\overline{\mho}_{\pi}\left(\underline{\eth}\right) = \left\{\tilde{\chi}_{\overline{\mho}}\left(\underline{\eth}\right) + i\tilde{Z}_{\overline{\mho}}\left(\underline{\eth}\right)\right\}$, is the subset of complex values, and also both $\tilde{\chi}_{\overline{\mho}}\left(\underline{\eth}\right), \tilde{Z}_{\overline{\mho}}\left(\underline{\eth}\right)$ belongs to the unit square in the complex plane and $\overline{\mho}_{\pi}\left(\underline{\eth}\right)$ demonstrating the truth grade for $\underline{\eth} \in \overleftarrow{\mathbb{X}}$. We use $\overline{\mho}_{\pi}\left(\underline{\eth}\right) = \overline{\mho}$ as a CHF element (CHFE), and to C as the set of all HFEs.

Definition 2: [35] For given two $CHFEs$ $\overline{\mho}_1$and $\overline{\mho}_2$, their fundamental operations are defined and denoted as follows:

1. $\overline{\mho}_1^{\,c} = \cup_{\tilde{\chi}_{\overline{\mho}_1}, \tilde{Z}_{\overline{\mho}_1} \epsilon \overline{\mho}_1} \left\{\left(1 - \tilde{\chi}_{\overline{\mho}_1}\right) + i\left(1 - \tilde{\chi}_{\overline{\mho}_1}\right)\right\}$
2. $\overline{\mho}_1 \cup \overline{\mho}_2 = \cup_{\tilde{\chi}_{\overline{\mho}_1}, \tilde{Z}_{\overline{\mho}_1} \epsilon \overline{\mho}_1, \tilde{\chi}_{\overline{\mho}_2}, \tilde{Z}_{\overline{\mho}_2} \epsilon \overline{\mho}_2} \left\{max\left(\tilde{\chi}_{\overline{\mho}_1}, \tilde{\chi}_{\overline{\mho}_2}\right) + i max\left(\tilde{Z}_{\overline{\mho}_1}, \tilde{Z}_{\overline{\mho}_2}\right)\right\}$
3. $\overline{\mho}_1 \cap \overline{\mho}_2 = \cup_{\tilde{\chi}_{\overline{\mho}_1}, \tilde{Z}_{\overline{\mho}_1} \epsilon \overline{\mho}_1, \tilde{\chi}_{\overline{\mho}_2}, \tilde{Z}_{\overline{\mho}_2} \epsilon \overline{\mho}_2} \left\{min\left(\tilde{\chi}_{\overline{\mho}_1}, \tilde{\chi}_{\overline{\mho}_2}\right) + i min\left(\tilde{Z}_{\overline{\mho}_1}, \tilde{Z}_{\overline{\mho}_2}\right)\right\}$

Definition 3: [27] Let λ be a parameter greater than zero $\underline{\eth}, \underline{\varsigma} \epsilon [0, 1]$, the Hamacher t-norm and t-conorm are revealed as

$$\Gamma_{\lambda}^{H}\left(\underline{\eth}, \underline{\varsigma}\right) = \frac{\underline{\eth}.\underline{\varsigma}}{\lambda + (1 - \lambda)\left(\underline{\eth} + \underline{\varsigma} - \underline{\eth}.\underline{\varsigma}\right)}; \lambda > 0,$$

and

$$S_{\lambda}^{H}\left(\underline{\eth}.\underline{\varsigma}\right) = \frac{\underline{\eth} + \underline{\varsigma} - \underline{\eth}.\underline{\varsigma} - (1 - \lambda)\underline{\eth}.\underline{\varsigma}}{\lambda + (1 - \lambda)\left(\underline{\eth} + \underline{\varsigma} - \underline{\eth}.\underline{\varsigma}\right)}, \lambda > 0$$

Definition 4: For a $CHFEs$ $\overline{\mho}$, the score function is defined and denoted as:

$$\mathfrak{s}(\overline{\mho}) = \frac{1}{\nabla(\overline{\mho})} \sum_{\tilde{\chi}_{\overline{\mho}}, \tilde{Z}_{\overline{\mho}} \epsilon \overline{\mho}} \left(\tilde{\chi} + i\tilde{Z}\right)$$

where $\nabla(\overline{\mho})$, total number of elements in $\overline{\mho}$. Ranking of two $CHFEs$, relates to the score function, if $\mathfrak{s}(\overline{\mho}_1) > \mathfrak{s}(\overline{\mho}_2)$, then$\overline{\mho}_1 > \overline{\mho}_2$; if $\mathfrak{s}(\overline{\mho}_1) = \mathfrak{s}(\overline{\mho}_2)$, then $\overline{\mho}_1 = \overline{\mho}_2$.

Definition 5: For $\overline{\mho}, \overline{\mho}_1$ and $\overline{\mho}_2$,are three $CHFEs$ and parameter $\lambda > 0$, the basic operations on CHFEs by employing the Hamacher t-norm and t-conorm, are devised as

1. $\overline{\mho}_1 \oplus_{\mathrm{H}} \overline{\mho}_2 =$
$\cup_{\tilde{\chi}_{\overline{\mho}_1}, \tilde{Z}_{\overline{\mho}_1} \epsilon \overline{\mho}_1, \tilde{\chi}_{\overline{\mho}_2}, \tilde{Z}_{\overline{\mho}_2} \epsilon \overline{\mho}_2} \left\{\left(\frac{\tilde{\chi}_{\overline{\mho}_1} + \tilde{\chi}_{\overline{\mho}_2} - \tilde{\chi}_{\overline{\mho}_1}.\tilde{\chi}_{\overline{\mho}_2} - (1-\lambda)\tilde{\chi}_{\overline{\mho}_1}.\tilde{\chi}_{\overline{\mho}_2}}{1 - (1-\lambda)\tilde{\chi}_{\overline{\mho}_1}.\tilde{\chi}_{\overline{\mho}_2}}\right) + i\left(\frac{\tilde{Z}_{\overline{\mho}_1} + \tilde{Z}_{\overline{\mho}_2} - \tilde{Z}_{\overline{\mho}_1}.\tilde{Z}_{\overline{\mho}_2} - (1-\lambda)\tilde{Z}_{\overline{\mho}_1}.\tilde{Z}_{\overline{\mho}_2}}{1 - (1-\lambda)\tilde{Z}_{\overline{\mho}_1}.\tilde{Z}_{\overline{\mho}_2}}\right)\right\}$

2. $\overline{\mho}_1 \otimes_{\mathrm{H}} \overline{\mho}_2 =$

$$\cup_{\tilde{\chi}_{\overline{\mho}_1},\tilde{Z}_{\overline{\mho}_1}\epsilon\overline{\mho}_1,\tilde{\chi}_{\overline{\mho}_2},\tilde{Z}_{\overline{\mho}_2}\epsilon\overline{\mho}_2}\left\{\left(\frac{\tilde{\chi}_{\overline{\mho}_1}\cdot\tilde{\chi}_{\overline{\mho}_2}}{\gimel+(1-\gimel)\left(\tilde{\chi}_{\overline{\mho}_1}+\tilde{\chi}_{\overline{\mho}_2}-\tilde{\chi}_{\overline{\mho}_1}\cdot\tilde{\chi}_{\overline{\mho}_2}\right)}\right)+i\left(\frac{\tilde{Z}_{\overline{\mho}_1}\cdot\tilde{Z}_{\overline{\mho}_2}}{\gimel+(1-\gimel)\left(\tilde{Z}_{\overline{\mho}_1}+\tilde{Z}_{\overline{\mho}_2}-\tilde{Z}_{\overline{\mho}_1}\cdot\tilde{Z}_{\overline{\mho}_2}\right)}\right)\right\}$$

3. $$\daleth\overline{\mho} = \cup_{\tilde{\chi}_{\overline{\mho}},\tilde{Z}_{\overline{\mho}}\epsilon\overline{\mho}}\left\{\left(\frac{\left(1+(\gimel-1)\tilde{\chi}_{\overline{\mho}}\right)^{\daleth}-\left(1-\tilde{\chi}_{\overline{\mho}}\right)^{\daleth}}{\left(1+(\gimel-1)\tilde{\chi}_{\overline{\mho}}\right)^{\daleth}+(\gimel-1)\left(1-\tilde{\chi}_{\overline{\mho}}\right)^{\daleth}}\right)+i\left(\frac{\left(1+(\gimel-1)\tilde{Z}_{\overline{\mho}}\right)^{\daleth}-\left(1-\tilde{Z}_{\overline{\mho}}\right)^{\daleth}}{\left(1+(\gimel-1)\tilde{Z}_{\overline{\mho}}\right)^{\daleth}+(\gimel-1)\left(1-\tilde{Z}_{\overline{\mho}}\right)^{\daleth}}\right)\right\},\quad \daleth>0$$

4. $$\overline{\mho}^{\daleth} = \cup_{\tilde{\chi}_{\overline{\mho}},\tilde{Z}_{\overline{\mho}}\epsilon\overline{\mho}}\left\{\left(\frac{\gimel\left(\tilde{\chi}_{\overline{\mho}}\right)^{\daleth}}{\left(1+(\gimel-1)\left(1-\tilde{\chi}_{\overline{\mho}}\right)\right)^{\daleth}+(\gimel-1)\left(\tilde{\chi}_{\overline{\mho}}\right)^{\daleth}}\right)+i\left(\frac{\gimel\left(\tilde{Z}_{\overline{\mho}}\right)^{\daleth}}{\left(1+(\gimel-1)\left(1-\tilde{Z}_{\overline{\mho}}\right)\right)^{\daleth}+(\gimel-1)\left(\tilde{Z}_{\overline{\mho}}\right)^{\daleth}}\right)\right\},\daleth>0$$

Proposition 1. For the three $CHFEs$, we have followings results:

1. $\overline{\mho}_1{}^{c\cdot}\otimes_{\mathrm{H}}\overline{\mho}_1{}^{c\cdot} = (\overline{\mho}_1\otimes_{\mathrm{H}}\overline{\mho}_2)^{c\cdot}$
2. $\daleth(\overline{\mho}^{c\cdot}) = (\overline{\mho}^{\daleth})^{c\cdot}, \daleth>0$
3. $(\overline{\mho}^{c\cdot})^{\daleth} = (\daleth\overline{\mho})^{c\cdot}, \daleth>0;$
4. $\overline{\mho}_1{}^{c\cdot}\oplus_{\mathrm{H}}\overline{\mho}_1{}^{c\cdot} = (\overline{\mho}_1\oplus_{\mathrm{H}}\overline{\mho}_2)^{c\cdot}$

Proof:

1. $\overline{\mho}_1{}^{c\cdot}\oplus_{\mathrm{H}}\overline{\mho}_1{}^{c\cdot} =$

$$\cup_{\tilde{\chi}_{\overline{\mho}_1},\tilde{Z}_{\overline{\mho}_1}\epsilon\overline{\mho}_1,\tilde{\chi}_{\overline{\mho}_2},\tilde{Z}_{\overline{\mho}_2}\epsilon\overline{\mho}_2}\left\{\left(\frac{\begin{array}{c}\left(\left(1-\tilde{\chi}_{\overline{\mho}_1}\right)+\left(1-\tilde{\chi}_{\overline{\mho}_2}\right)-\left(1-\tilde{\chi}_{\overline{\mho}_1}\right).\left(1-\tilde{\chi}_{\overline{\mho}_2}\right)\right)\\-\left((1-\gimel)\left(1-\tilde{\chi}_{\overline{\mho}_1}\right).\left(1-\tilde{\chi}_{\overline{\mho}_2}\right)\right)\end{array}}{1-(1-\gimel)\left(1-\tilde{\chi}_{\overline{\mho}_1}\right).\left(1-\tilde{\chi}_{\overline{\mho}_2}\right)}\right)+i\left(\frac{\begin{array}{c}\left(\left(1-\tilde{Z}_{\overline{\mho}_1}\right)+\left(1-\tilde{Z}_{\overline{\mho}_2}\right)-\left(1-\tilde{Z}_{\overline{\mho}_1}\right).\left(1-\tilde{Z}_{\overline{\mho}_2}\right)\right)\\-\left((1-\gimel)\left(1-\tilde{Z}_{\overline{\mho}_1}\right).\left(1-\tilde{Z}_{\overline{\mho}_2}\right)\right)\end{array}}{1-(1-\gimel)\left(1-\tilde{Z}_{\overline{\mho}_1}\right).\left(1-\tilde{Z}_{\overline{\mho}_2}\right)}\right)\right\}$$

$$= \bigcup_{\tilde{\chi}_{\overline{\mho}_1},\tilde{Z}_{\overline{\mho}_1}\epsilon\overline{\mho}_1,\tilde{\chi}_{\overline{\mho}_2},\tilde{Z}_{\overline{\mho}_2}\epsilon\overline{\mho}_2}\left\{\begin{array}{l}\left(\dfrac{\tilde{\chi}_{\overline{\mho}_1}+\tilde{\chi}_{\overline{\mho}_2}+(\gimel)\left(1-\tilde{\chi}_{\overline{\mho}_1}\right)\left(1-\tilde{\chi}_{\overline{\mho}_1}\right)}{\tilde{\chi}_{\overline{\mho}_1}+\tilde{\chi}_{\overline{\mho}_2}-\tilde{\chi}_{\overline{\mho}_1}\cdot\tilde{\chi}_{\overline{\mho}_2}+(\gimel)\left(1-\tilde{\chi}_{\overline{\mho}_1}\right)\left(1-\tilde{\chi}_{\overline{\mho}_1}\right)}\right)\\+i\left(\dfrac{\tilde{Z}_{\overline{\mho}_1}+\tilde{Z}_{\overline{\mho}_2}+(\gimel)\left(1-\tilde{Z}_{\overline{\mho}_1}\right)\left(1-\tilde{Z}_{\overline{\mho}_2}\right)}{\tilde{Z}_{\overline{\mho}_1}+\tilde{Z}_{\overline{\mho}_2}-\tilde{Z}_{\overline{\mho}_1}.\tilde{Z}_{\overline{\mho}_2}+(\gimel)\left(1-\tilde{Z}_{\overline{\mho}_1}\right).\left(1-\tilde{Z}_{\overline{\mho}_2}\right)}\right)\end{array}\right\}$$

$$= \bigcup_{\tilde{\chi}_{\overline{\mho}_1},\tilde{Z}_{\overline{\mho}_1}\epsilon\overline{\mho}_1,\tilde{\chi}_{\overline{\mho}_2},\tilde{Z}_{\overline{\mho}_2}\epsilon\overline{\mho}_2}\left\{\begin{array}{l}\left(1-\left(\dfrac{\tilde{\chi}_{\overline{\mho}_1}\cdot\tilde{\chi}_{\overline{\mho}_2}}{\gimel+(1-\gimel)\left(\tilde{\chi}_{\overline{\mho}_1}+\tilde{\chi}_{\overline{\mho}_2}-\tilde{\chi}_{\overline{\mho}_1}\cdot\tilde{\chi}_{\overline{\mho}_2}\right)}\right)\right)\\+i\left(1-\left(\dfrac{\tilde{Z}_{\overline{\mho}_1}.\tilde{Z}_{\overline{\mho}_2}}{\gimel+(1-\gimel)\left(\tilde{Z}_{\overline{\mho}_1}+\tilde{Z}_{\overline{\mho}_2}-\tilde{Z}_{\overline{\mho}_1}.\tilde{Z}_{\overline{\mho}_2}\right)}\right)\right)\end{array}\right\}$$

$$= (\overline{\mho}_1\oplus_{\mathrm{H}}\overline{\mho}_2)^{c\cdot}$$

2. $$\overline{\mho}_1{}^{c\cdot}\otimes_{\mathrm{H}}\overline{\mho}_1{}^{c\cdot} = \cup_{\tilde{\chi}_{\overline{\mho}_1},\tilde{Z}_{\overline{\mho}_1}\epsilon\overline{\mho}_1,\tilde{\chi}_{\overline{\mho}_2},\tilde{Z}_{\overline{\mho}_2}\epsilon\overline{\mho}_2}\left\{\begin{array}{l}\left(\dfrac{\left(1-\tilde{\chi}_{\overline{\mho}_1}\right)\left(1-\tilde{\chi}_{\overline{\mho}_2}\right)}{\gimel+(1-\gimel)\left(1-\tilde{\chi}_{\overline{\mho}_1}\right)+\left(1-\tilde{\chi}_{\overline{\mho}_2}\right)-\left(1-\tilde{\chi}_{\overline{\mho}_1}\right)\left(1-\tilde{\chi}_{\overline{\mho}_2}\right)}\right)\\+i\left(\dfrac{\left(1-\tilde{Z}_{\overline{\mho}_1}\right)\left(1-\tilde{Z}_{\overline{\mho}_2}\right)}{\gimel+(1-\gimel)\left(1-\tilde{Z}_{\overline{\mho}_1}\right)+\left(1-\tilde{Z}_{\overline{\mho}_2}\right)-\left(1-\tilde{Z}_{\overline{\mho}_1}\right)\left(1-\tilde{Z}_{\overline{\mho}_2}\right)}\right)\end{array}\right\}$$

$$= \bigcup_{\tilde{\chi}_{\overline{\mho}_1},\tilde{Z}_{\overline{\mho}_1}\epsilon\overline{\mho}_1,\tilde{\chi}_{\overline{\mho}_2},\tilde{Z}_{\overline{\mho}_2}\epsilon\overline{\mho}_2}\left\{\left(\frac{1-\tilde{\chi}_{\overline{\mho}_1}-\tilde{\chi}_{\overline{\mho}_2}+\tilde{\chi}_{\overline{\mho}_1}\cdot\tilde{\chi}_{\overline{\mho}_2}}{1-(1-\gimel)\tilde{\chi}_{\overline{\mho}_1}\cdot\tilde{\chi}_{\overline{\mho}_2}}\right)+i\left(\frac{1-\tilde{Z}_{\overline{\mho}_1}-\tilde{Z}_{\overline{\mho}_2}+\tilde{Z}_{\overline{\mho}_1}.\tilde{Z}_{\overline{\mho}_2}}{1-(1-\gimel)\tilde{Z}_{\overline{\mho}_1}.\tilde{Z}_{\overline{\mho}_2}}\right)\right\}$$

$$= \bigcup_{\tilde{\chi}_{\overline{\mho}_1},\tilde{Z}_{\overline{\mho}_1}\epsilon\overline{\mho}_1,\tilde{\chi}_{\overline{\mho}_2},\tilde{Z}_{\overline{\mho}_2}\epsilon\overline{\mho}_2}\left\{\begin{array}{l}\left(1-\left(\dfrac{\tilde{\chi}_{\overline{\mho}_1}+\tilde{\chi}_{\overline{\mho}_2}-\tilde{\chi}_{\overline{\mho}_1}\cdot\tilde{\chi}_{\overline{\mho}_2}-(1-\gimel)\tilde{\chi}_{\overline{\mho}_1}\cdot\tilde{\chi}_{\overline{\mho}_2}}{1-(1-\gimel)\tilde{\chi}_{\overline{\mho}_1}\cdot\tilde{\chi}_{\overline{\mho}_2}}\right)\right)\\+i\left(1-\left(\dfrac{\tilde{Z}_{\overline{\mho}_1}+\tilde{Z}_{\overline{\mho}_2}-\tilde{Z}_{\overline{\mho}_1}.\tilde{Z}_{\overline{\mho}_2}-(1-\gimel)\tilde{Z}_{\overline{\mho}_1}.\tilde{Z}_{\overline{\mho}_2}}{1-(1-\gimel)\tilde{Z}_{\overline{\mho}_1}.\tilde{Z}_{\overline{\mho}_2}}\right)\right)\end{array}\right\}$$

$$= (\overline{\mho}_1 \otimes_{\mathrm{H}} \overline{\mho}_2)^{c^{\cdot}}$$

3. $$(\overline{\mho}^{c^{\cdot}}) = \cup_{\tilde{\chi}_{\overline{\mho}}, \tilde{Z}_{\overline{\mho}} \in \overline{\mho}} \left\{ \left(\frac{\left(1+(1-\gimel)(1-\tilde{\chi}_{\overline{\mho}})\right)^{\daleth} - (\tilde{\chi}_{\overline{\mho}})^{\daleth}}{\left(1+(1-\gimel)(1-\tilde{\chi}_{\overline{\mho}})\right)^{\daleth} + (1-\gimel)(\tilde{\chi}_{\overline{\mho}})^{\daleth}} \right) + i \left(\frac{\left(1+(1-\gimel)(1-\tilde{Z}_{\overline{\mho}})\right)^{\daleth} - (\tilde{Z}_{\overline{\mho}})^{\daleth}}{\left(1+(1-\gimel)(1-\tilde{Z}_{\overline{\mho}})\right)^{\daleth} + (1-\gimel)(\tilde{Z}_{\overline{\mho}})^{\daleth}} \right) \right\}$$

$$= \bigcup_{\tilde{\chi}_{\overline{\mho}}, \tilde{Z}_{\overline{\mho}} \in \overline{\mho}} \left\{ \begin{array}{l} \left(1 - \left(\frac{\gimel (\tilde{\chi}_{\overline{\mho}})^{\daleth}}{\left(1 + (1-\gimel)(1-\tilde{\chi}_{\overline{\mho}})\right)^{\daleth} + (1-\gimel)(\tilde{\chi}_{\overline{\mho}})^{\daleth}} \right) \right) \\ + i \left(1 - \left(\frac{\gimel (\tilde{Z}_{\overline{\mho}})^{\daleth}}{\left(1 + (1-\gimel)(1-\tilde{Z}_{\overline{\mho}})\right)^{\daleth} + (1-\gimel)(\tilde{Z}_{\overline{\mho}})^{\daleth}} \right) \right) \end{array} \right\}$$

$$= (\overline{\mho}^{\daleth})^{c^{\cdot}}$$

4. $$(\overline{\mho}^{c^{\cdot}})^{\daleth} = \cup_{\tilde{\chi}_{\overline{\mho}}, \tilde{Z}_{\overline{\mho}} \in \overline{\mho}} \left\{ \left(\frac{\gimel (1-\tilde{\chi}_{\overline{\mho}})^{\daleth}}{\left(1+(1-\gimel)(\tilde{\chi}_{\overline{\mho}})\right)^{\daleth} + (1-\gimel)(1-\tilde{\chi}_{\overline{\mho}})^{\daleth}} \right) + i \left(\frac{\gimel (1-\tilde{Z}_{\overline{\mho}})^{\daleth}}{\left(1+(1-\gimel)(\tilde{Z}_{\overline{\mho}})\right)^{\daleth} + (1-\gimel)(1-\tilde{Z}_{\overline{\mho}})^{\daleth}} \right) \right\}$$

$$= \bigcup_{\tilde{\chi}_{\overline{\mho}}, \tilde{Z}_{\overline{\mho}} \in \overline{\mho}} \left\{ \begin{array}{l} \left(1 - \frac{\left(1 + (1-\gimel)(\tilde{\chi}_{\overline{\mho}})\right)^{\daleth} - (1-\tilde{\chi}_{\overline{\mho}})^{\daleth}}{\left(1 + (1-\gimel)(\tilde{\chi}_{\overline{\mho}})\right)^{\daleth} + (1-\gimel)(1-\tilde{\chi}_{\overline{\mho}})^{\daleth}} \right) \\ + i \left(1 - \frac{\left(1 + (1-\gimel)\tilde{Z}_{\overline{\mho}}\right)^{\daleth} - \left(1-\tilde{Z}_{\overline{\mho}}\right)^{\daleth}}{\left(1 + (1-\gimel)\tilde{Z}_{\overline{\mho}}\right)^{\daleth} + (1-\gimel)\left(1-\tilde{Z}_{\overline{\mho}}\right)^{\daleth}} \right) \end{array} \right\}$$

$$= (\daleth \overline{\mho})^{c^{\cdot}}$$

3. Hamacher AOs within Complex Hesitant Fuzzy framework

In this section, we interpret CHFHWA, GCHFHWA, CHFHWG and GCHFHWG operators.

Definition 6: Let $\pi = \{\overline{\mho}_1, \overline{\mho}_2, \dots, \overline{\mho}_{^\circ\mathrm{F}}\}$, be the set of $CHFEs$ such that its cardinality is equal to °F, ƒ is a mapping on π, ƒ: → [0, 1], then

$$\mathfrak{f}_{\pi} = \cup_{\tilde{\chi}_{\overline{\mho}}, \tilde{Z}_{\overline{\mho}} \in \{\overline{\mho}_1 \times \overline{\mho}_2 \times \dots \times \overline{\mho}_{^\circ\mathrm{F}}\}} \{\mathfrak{f}(\tilde{\chi})\}.$$

Definition 7: Suppose π be a set of n $CHFEs$, such that $\pi = (\overline{\mho}_1, \overline{\mho}_2, \dots, \overline{\mho}_{^\circ\mathrm{F}})$. An $CHF\mathrm{H}WA$ operator is a function $C^{^\circ\mathrm{F}} \longrightarrow C$ Such that

$$CHF\mathrm{H}WA_{\gimel}(\overline{\mho}_1, \overline{\mho}_2, \dots, \overline{\mho}_{^\circ\mathrm{F}}) = \omega_1 \overline{\mho}_1 \oplus_H \omega_2 \overline{\mho}_2 \oplus_H \dots . \oplus_H \underline{\omega}_{^\circ\mathrm{F}} \overline{\mho}_{^\circ\mathrm{F}} = \overset{^\circ\mathrm{F}}{\underset{\zeta=1}{\oplus_H}} \underline{\omega}_{\zeta} \overline{\mho}_{\zeta},$$

Where the parameter $\gimel$, should be a positive number with the weight vector $\underline{\omega} = \left(\underline{\omega}_1, \underline{\omega}_2, \dots, \underline{\omega}_{^\circ\mathrm{F}}\right)^{\mathcal{T}}$, whose each weight for each argument belongs to unit interval and the sum of all weights must be equal to the one $\left(\sum_{\mathfrak{i}=1}^{^\circ\mathrm{F}} \underline{\omega}_{\mathfrak{i}} = 1\right)$.

Theorem 1. Let $\pi = (\overline{\mho}_1, \overline{\mho}_2, \dots, \overline{\mho}_{^\circ\mathrm{F}})$, consist of $CHFEs$, nonetheless the aggregated outcome by employing $CHF\mathrm{H}WA$ operator is a $CHFN$, and

$$CHFHWA_{\lambda}(\overline{\mho}_1,\overline{\mho}_2,\dots,\overline{\mho}_{°F}) = \bigcup_{\tilde{\chi}_{\overline{\mho}_1},\tilde{Z}_{\overline{\mho}_1}\epsilon\overline{\mho}_1,\dots,\tilde{\chi}_{\overline{\mho}_{°F}},\tilde{Z}_{\overline{\mho}_{°F}}\epsilon\overline{\mho}_{°F}} \left\{ \begin{array}{l} \left(\frac{\prod_{\zeta=1}^{°F}\left(1+(\lambda-1)\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\omega_\zeta}-\prod_{\zeta=1}^{°F}\left(1-\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\omega_\zeta}}{\prod_{\zeta=1}^{°F}\left(1+(\lambda-1)\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\omega_\zeta}+(\lambda-1)\prod_{\zeta=1}^{°F}\left(1-\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\omega_\zeta}}\right) \\ +i\left(\frac{\prod_{\zeta=1}^{°F}\left(1+(\lambda-1)\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\omega_\zeta}-\prod_{\zeta=1}^{°F}\left(1-\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\omega_\zeta}}{\prod_{\zeta=1}^{°F}\left(1+(\lambda-1)\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\omega_\zeta}+(\lambda-1)\prod_{\zeta=1}^{°F}\left(1-\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\omega_\zeta}}\right) \end{array} \right\} \tag{1}$$

Proof: We will proceed this proof by using mathematical induction on °F, as follows:
Clearly true for $°F = 1$.
Let us suppose Eq. (1) stands for $°F = \dot{k}$, that is

$$CHFHWA_{\lambda}(\overline{\mho}_1,\overline{\mho}_2,\dots,\overline{\mho}_{\dot{k}}) = \bigcup_{\tilde{\chi}_{\overline{\mho}_1},\tilde{Z}_{\overline{\mho}_1}\epsilon\overline{\mho}_1,\dots,\tilde{\chi}_{\overline{\mho}_{\dot{k}}},\tilde{Z}_{\overline{\mho}_{\dot{k}}}\epsilon\overline{\mho}_{\dot{k}}} \left\{\left(\frac{\prod_{\zeta=1}^{\dot{k}}\left(1+(\lambda-1)\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\omega_\zeta}-\prod_{\zeta=1}^{\dot{k}}\left(1-\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\omega_\zeta}}{\prod_{\zeta=1}^{\dot{k}}\left(1+(\lambda-1)\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\omega_\zeta}+(\lambda-1)\prod_{\zeta=1}^{\dot{k}}\left(1-\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\omega_\zeta}}\right)+i\left(\frac{\prod_{\zeta=1}^{\dot{k}}\left(1+(\lambda-1)\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\omega_\zeta}-\prod_{\zeta=1}^{\dot{k}}\left(1-\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\omega_\zeta}}{\prod_{\zeta=1}^{\dot{k}}\left(1+(\lambda-1)\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\omega_\zeta}+(\lambda-1)\prod_{\zeta=1}^{\dot{k}}\left(1-\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\omega_\zeta}}\right)\right\}$$

Now for $°F = \dot{k}+1$, from Definition 6 and Definition 4. We get

$$\begin{aligned} CHFHWA_{\lambda}(\overline{\mho}_1,\overline{\mho}_2,\dots,\overline{\mho}_{\dot{k}+1}) &= \omega_1\overline{\mho}_1\oplus_H\omega_2\overline{\mho}_2\oplus_H\dots\oplus_H\underline{\omega}_{\dot{k}}\overline{\mho}_{\dot{k}}\oplus_H\underline{\omega}_{\dot{k}+1}\overline{\mho}_{\dot{k}+1} \\ &= CHFHWA_{\lambda}(\overline{\mho}_1,\overline{\mho}_2,\dots,\overline{\mho}_{\dot{k}})\oplus_H\underline{\omega}_{\dot{k}+1}\overline{\mho}_{\dot{k}+1} \\ &= \bigcup_{\tilde{\chi}_{\overline{\mho}_1},\tilde{Z}_{\overline{\mho}_1}\epsilon\overline{\mho}_1,\dots,\tilde{\chi}_{\overline{\mho}_{\dot{k}}},\tilde{Z}_{\overline{\mho}_{\dot{k}}}\epsilon\overline{\mho}_{\dot{k}}} \left\{ \begin{array}{l} \left(\frac{\prod_{\zeta=1}^{\dot{k}}\left(1+(\lambda-1)\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\omega_\zeta}-\prod_{\zeta=1}^{\dot{k}}\left(1-\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\omega_\zeta}}{\prod_{\zeta=1}^{\dot{k}}\left(1+(\lambda-1)\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\omega_\zeta}+(\lambda-1)\prod_{\zeta=1}^{\dot{k}}\left(1-\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\omega_\zeta}}\right) \\ +i\left(\frac{\prod_{\zeta=1}^{\dot{k}}\left(1+(\lambda-1)\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\omega_\zeta}-\prod_{\zeta=1}^{\dot{k}}\left(1-\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\omega_\zeta}}{\prod_{\zeta=1}^{\dot{k}}\left(1+(\lambda-1)\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\omega_\zeta}+(\lambda-1)\prod_{\zeta=1}^{\dot{k}}\left(1-\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\omega_\zeta}}\right) \end{array} \right\} \oplus_H \\ & \bigcup_{\tilde{\chi}_{\overline{\mho}_1},\tilde{Z}_{\overline{\mho}_1}\epsilon\overline{\mho}_1,\dots,\tilde{\chi}_{\overline{\mho}_{\dot{k}+1}},\tilde{Z}_{\overline{\mho}_{\dot{k}+1}}\epsilon\overline{\mho}_{\dot{k}+1}} \left\{ \begin{array}{l} \left(\frac{\prod_{j=1}^{\dot{k}+1}\left(1+(\lambda-1)\tilde{\chi}_{\overline{\mho}_{\dot{k}+1}}\right)^{\underline{\omega}_{\dot{k}+1}}-\prod_{j=1}^{\dot{k}}\left(1-\tilde{\chi}_{\overline{\mho}_{\dot{k}+1}}\right)^{\underline{\omega}_{\dot{k}+1}}}{\prod_{j=1}^{\dot{k}+1}\left(1+(\lambda-1)\tilde{\chi}_{\overline{\mho}_{\dot{k}+1}}\right)^{\underline{\omega}_{\dot{k}+1}}+(\lambda-1)\prod_{j=1}^{\dot{k}+1}\left(1-\tilde{\chi}_{\overline{\mho}_{\dot{k}+1}}\right)^{\underline{\omega}_{\dot{k}+1}}}\right) \\ +i\left(\frac{\prod_{j=1}^{\dot{k}+1}\left(1+(\lambda-1)\tilde{Z}_{\overline{\mho}_{\dot{k}+1}}\right)^{\underline{\omega}_{\dot{k}+1}}-\prod_{j=1}^{\dot{k}}\left(1-\tilde{Z}_{\overline{\mho}_{\dot{k}+1}}\right)^{\underline{\omega}_{\dot{k}+1}}}{\prod_{j=1}^{\dot{k}+1}\left(1+(\lambda-1)\tilde{Z}_{\overline{\mho}_{\dot{k}+1}}\right)^{\underline{\omega}_{\dot{k}+1}}+(\lambda-1)\prod_{j=1}^{\dot{k}+1}\left(1-\tilde{Z}_{\overline{\mho}_{\dot{k}+1}}\right)^{\underline{\omega}_{\dot{k}+1}}}\right) \end{array} \right\} \end{aligned}$$

Let $\beta_1 = \prod_{\zeta=1}^{\dot{k}}\left(1+(\lambda-1)\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\omega_\zeta}$, $\delta_1 = \prod_{\zeta=1}^{\dot{k}}\left(1-\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\omega_\zeta}$, $\beta_2 = \prod_{\zeta=1}^{\dot{k}+1}\left(1+(\lambda-1)\tilde{\chi}_{\overline{\mho}_{\dot{k}+1}}\right)^{\underline{\omega}_{\dot{k}+1}}$ and $\delta_2 = \prod_{\zeta=1}^{\dot{k}}\left(1-\tilde{\chi}_{\overline{\mho}_{\dot{k}+1}}\right)^{\underline{\omega}_{\dot{k}+1}}$

$$\vartheta_1 = \prod_{\zeta=1}^{\dot{k}}\left(1+(\lambda-1)\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\omega_\zeta}, \rho_1 = \prod_{\zeta=1}^{\dot{k}}\left(1-\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\omega_\zeta},$$

$$\vartheta_2 = \prod_{\zeta=1}^{\dot{k}+1}\left(1+(\lambda-1)\tilde{Z}_{\overline{\mho}_{\dot{k}+1}}\right)^{\underline{\omega}_{\dot{k}+1}}, \rho_2 = \prod_{\zeta=1}^{\dot{k}}\left(1-\tilde{Z}_{\overline{\mho}_{\dot{k}+1}}\right)^{\underline{\omega}_{\dot{k}+1}}$$

Then Eq. (1) becomes

$CHFHWA_{\lambda}(\overline{\mho}_1,\overline{\mho}_2,\dots,\overline{\mho}_{\dot{k}+1}) = \frac{\beta_1-\delta_1}{\beta_1+(\lambda-1)\delta_1} + i\frac{\vartheta_1-\rho_1}{\vartheta_1+(\lambda-1)\rho_1}$, and $\underline{\omega}_{\dot{k}+1}\overline{\mho}_{\dot{k}+1} = \frac{\beta_2-\delta_2}{\beta_2+(\lambda-1)\delta_2} + i\frac{\vartheta_2-\rho_2}{\vartheta_2+(\lambda-1)\rho_2}$.

By Definition 4, we get $CHF\mathrm{H}WA_{\lambda}(\overline{\mho}_1,\overline{\mho}_2,\dots,\overline{\mho}_{k})\oplus_H \underline{\omega}_{k+1}\overline{\mho}_{k+1} = \frac{\beta_1-\delta_1}{\beta_1+(\lambda-1)\delta_1} + i\frac{\vartheta_1-\rho_1}{\vartheta_1+(\lambda-1)\rho_1} \oplus_H \frac{\beta_2-\delta_2}{\beta_2+(\lambda-1)\delta_2} + i\frac{\vartheta_2-\rho_2}{\vartheta_2+(\lambda-1)\rho_2}$

$$= \bigcup_{\tilde{\chi}_{\overline{\mho}_1},\tilde{Z}_{\overline{\mho}_1}\epsilon\overline{\mho}_1,\dots,\tilde{\chi}_{\overline{\mho}_{k}},\tilde{Z}_{\overline{\mho}_{k}}\epsilon\overline{\mho}_{k}} \left\{ \frac{\left(\begin{array}{c} \frac{\beta_1-\delta_1}{\beta_1+(\lambda-1)\delta_1} + i\frac{\vartheta_1-\rho_1}{\vartheta_1+(\lambda-1)\rho_1} \\ +\frac{\beta_2-\delta_2}{\beta_2+(\lambda-1)\delta_2} + i\frac{\vartheta_2-\rho_2}{\vartheta_2+(\lambda-1)\rho_2} - \\ \frac{\beta_1-\delta_1}{\beta_1+(\lambda-1)\delta_1} + i\frac{\vartheta_1-\rho_1}{\vartheta_1+(\lambda-1)\rho_1} \\ \times\frac{\beta_2-\delta_2}{\beta_2+(\lambda-1)\delta_2} + i\frac{\vartheta_2-\rho_2}{\vartheta_2+(\lambda-1)\rho_2} - \\ (\lambda-1)\frac{\beta_1-\delta_1}{\beta_1+(\lambda-1)\delta_1} + i\frac{\vartheta_1-\rho_1}{\vartheta_1+(\lambda-1)\rho_1} \\ \times\frac{\beta_2-\delta_2}{\beta_2+(\lambda-1)\delta_2} + i\frac{\vartheta_2-\rho_2}{\vartheta_2+(\lambda-1)\rho_2} \end{array}\right)}{\left(\begin{array}{c} 1-(\lambda-1)\frac{\beta_1-\delta_1}{\beta_1+(\lambda-1)\delta_1} + i\frac{\vartheta_1-\rho_1}{\vartheta_1+(\lambda-1)\rho_1} \\ \times\frac{\beta_2-\delta_2}{\beta_2+(\lambda-1)\delta_2} + i\frac{\vartheta_2-\rho_2}{\vartheta_2+(\lambda-1)\rho_2} \end{array}\right)} \right\}$$

$$= \bigcup_{\tilde{\chi}_{\overline{\mho}_1},\tilde{Z}_{\overline{\mho}_1}\epsilon\overline{\mho}_1,\dots,\tilde{\chi}_{\overline{\mho}_{k+1}},\tilde{Z}_{\overline{\mho}_{k}}\epsilon\overline{\mho}_{k+1}} \left\{ \begin{array}{c} \frac{\prod_{\zeta=1}^{k+1}\left(1+(\lambda-1)\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_j} - \prod_{j=1}^{k+1}\left(1-\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta}}{\prod_{\zeta=1}^{k+1}\left(1+(\lambda-1)\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_j} + (\lambda-1)\prod_{j=1}^{k+1}\left(1-\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta}} \\ +i\frac{\prod_{\zeta=1}^{k+1}\left(1+(\lambda-1)\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_j} - \prod_{j=1}^{k+1}\left(1-\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta}}{\prod_{\zeta=1}^{k+1}\left(1+(\lambda-1)\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_j} + (\lambda-1)\prod_{j=1}^{k+1}\left(1-\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta}} \end{array} \right\}$$

From above it is clear that Eq. (1), stands for $°\mathrm{F} = k+1$. Thus Eq. (1), stands for all °F.

Remark 1. By letting $\lambda = 1$, the $CHF\mathrm{H}WA$ operator will be converted in the shape, given below

$$CHFWA_{\underline{\omega}}(\overline{\mho}_1,\overline{\mho}_2,\dots,\overline{\mho}_{°\mathrm{F}}) = \underset{\zeta=1}{\overset{°\mathrm{F}}{\oplus}} \underline{\omega}_\zeta\overline{\mho}_\zeta$$

$$= \bigcup_{\tilde{\chi}_{\overline{\mho}_1},\tilde{Z}_{\overline{\mho}_1}\epsilon\overline{\mho}_1,\dots,\tilde{\chi}_{\overline{\mho}_{°\mathrm{F}}},\tilde{Z}_{\overline{\mho}_{°\mathrm{F}}}\epsilon\overline{\mho}_{°\mathrm{F}}} \left\{\left(1-\prod_{\zeta=1}^{°\mathrm{F}}\left(1-\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta}\right) + i\left(1-\prod_{\zeta=1}^{°\mathrm{F}}\left(1-\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta}\right)\right\}$$

which is named as CHF weighted averaging ($CHF\mathrm{H}WA$) operator. If $\lambda = 2$, the $CHF\mathrm{H}WA$ operator will be is converted in the shape below

$$CHFEWA_{\underline{\omega}}(\overline{\mho}_1,\overline{\mho}_2,\dots,\overline{\mho}_{°\mathrm{F}}) = \underset{\zeta=1}{\overset{°\mathrm{F}}{\oplus_\varepsilon}} \underline{\omega}_\zeta\overline{\mho}_\zeta$$

$$= \bigcup_{\tilde{\chi}_{\overline{\mho}_1},\tilde{Z}_{\overline{\mho}_1}\epsilon\overline{\mho}_1,\dots,\tilde{\chi}_{\overline{\mho}_{°\mathrm{F}}},\tilde{Z}_{\overline{\mho}_{°\mathrm{F}}}\epsilon\overline{\mho}_{°\mathrm{F}}} \left\{\left(\frac{\prod_{\zeta=1}^{°\mathrm{F}}\left(1+\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta} - \prod_{\zeta=1}^{°\mathrm{F}}\left(1-\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta}}{\prod_{\zeta=1}^{°\mathrm{F}}\left(1+\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta} + \prod_{\zeta=1}^{°\mathrm{F}}\left(1-\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta}}\right) + i\left(\frac{\prod_{\zeta=1}^{°\mathrm{F}}\left(1+\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta} - \prod_{\zeta=1}^{°\mathrm{F}}\left(1-\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta}}{\prod_{\zeta=1}^{°\mathrm{F}}\left(1+\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta} + \prod_{\zeta=1}^{°\mathrm{F}}\left(1-\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta}}\right)\right\},$$

Which is named as CHF Einstein weighted averaging ($CHFEWA$) operator. Moreover, if $\underline{\omega} = \left(\frac{1}{°\mathrm{F}},\frac{1}{°\mathrm{F}},\dots,\frac{1}{°\mathrm{F}}\right)^{T}$, then the $CHF\mathrm{H}WA$ operator will be converted in the shape, given below

$$CHFHA(\overline{\mho}_1,\overline{\mho}_2,\dots,\overline{\mho}_{{}^{\circ}F}) = \bigcup_{\tilde{\chi}_{\overline{\mho}_1},\tilde{Z}_{\overline{\mho}_1}\epsilon\overline{\mho}_1,\dots,\tilde{\chi}_{\overline{\mho}_{{}^{\circ}F}},\tilde{Z}_{\overline{\mho}_{{}^{\circ}F}}\epsilon\overline{\mho}_{{}^{\circ}F}} \left\{ \begin{array}{l} \left(\dfrac{\prod_{\zeta=1}^{{}^{\circ}F}\left(1+(\lambda-1)\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\frac{1}{{}^{\circ}F}} - \prod_{\zeta=1}^{{}^{\circ}F}\left(1-\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\frac{1}{{}^{\circ}F}}}{\prod_{\zeta=1}^{\ddot{n}}\left(1+(\lambda-1)\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\frac{1}{{}^{\circ}F}} + (\lambda-1)\prod_{\zeta=1}^{{}^{\circ}F}\left(1-\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\frac{1}{{}^{\circ}F}}} \right) \\ +i\left(\dfrac{\prod_{\zeta=1}^{{}^{\circ}F}\left(1+(\lambda-1)\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\frac{1}{{}^{\circ}F}} - \prod_{\zeta=1}^{{}^{\circ}F}\left(1-\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\frac{1}{{}^{\circ}F}}}{\prod_{\zeta=1}^{\ddot{n}}\left(1+(\lambda-1)\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\frac{1}{{}^{\circ}F}} + (\lambda-1)\prod_{\zeta=1}^{{}^{\circ}F}\left(1-\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\frac{1}{{}^{\circ}F}}} \right) \end{array} \right\}$$

Named as CHF Hamacher averaging ($CHFHA$) operator.

Lemma 1. Let $\underline{\delta}_\zeta > 0, \underline{\omega}_\zeta > 0, \zeta = 1,2,..,{}^{\circ}F$, and $\sum_{\zeta=1}^{{}^{\circ}F}\underline{\omega}_\zeta = 1$, then $\prod_{\zeta=1}^{{}^{\circ}F}\left(\underline{\delta}_\zeta\right)^{\underline{\omega}_\zeta} \le \sum_{\zeta=1}^{{}^{\circ}F}\underline{\omega}_\zeta\underline{\delta}_\zeta$, with equality if and only if $\underline{\delta}_1 = \underline{\delta}_2 =,\dots,\underline{\delta}_{{}^{\circ}F}$.

Proposition 2. Let us suppose $\pi = (\overline{\mho}_1,\overline{\mho}_2,\dots,\overline{\mho}_{{}^{\circ}F})$, be a set of ${}^{\circ}F$ $CHFEs$ and the weight vector $\underline{\omega}$ of $\overline{\mho}_\zeta(\zeta = 1,2\dots,{}^{\circ}F)$ is $\underline{\omega} = \left(\underline{\omega}_1,\underline{\omega}_2,\dots,\underline{\omega}_{{}^{\circ}F}\right)^{T}$. Whose weights each belong to the unit interval, and the sum of weights must be equal to one $\left(\sum_{\hat{\imath}=1}^{{}^{\circ}F}\underline{\omega}_{\hat{\imath}} = 1\right)$.Then $CHFHWA_\lambda(\overline{\mho}_1,\overline{\mho}_2,\dots,\overline{\mho}_{{}^{\circ}F}) \le CHFHA(\overline{\mho}_1,\overline{\mho}_2,\dots,\overline{\mho}_{{}^{\circ}F})$.

Proof: Let $\tilde{\chi}_{\overline{\mho}_1},\tilde{Z}_{\overline{\mho}_1}\epsilon\overline{\mho}_1,\dots,\tilde{\chi}_{\overline{\mho}_{{}^{\circ}F}},\tilde{Z}_{\overline{\mho}_{{}^{\circ}F}}\epsilon\overline{\mho}_{{}^{\circ}F}$, from previous Lemma 1. Form a well-known result if $\overline{\mho}_1$ and $\overline{\mho}_2$ are complex numbers then $\overline{\mho}_1 \le \overline{\mho}_2$ iff $\tilde{\chi}_{\overline{\mho}_1} \le \tilde{\chi}_{\overline{\mho}_2}$ and $\tilde{Z}_{\overline{\mho}_1} \le \tilde{Z}_{\overline{\mho}_2}$. We have

$$\prod_{\zeta=1}^{{}^{\circ}F}\left(1+(\lambda-1)\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta} + (\lambda-1)\prod_{\zeta=1}^{{}^{\circ}F}\left(1-\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta} \le \sum_{\zeta=1}^{{}^{\circ}F}\underline{\omega}_\zeta\left(1+(\lambda-1)\tilde{\chi}_{\overline{\mho}_\zeta}\right) + (\lambda-1)\sum_{\zeta=1}^{{}^{\circ}F}\underline{\omega}_\zeta\left(1+\tilde{\chi}_{\overline{\mho}_\zeta}\right)$$

Then

$$\frac{\prod_{\zeta=1}^{{}^{\circ}F}\left(1+(\lambda-1)\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta} - \prod_{\zeta=1}^{{}^{\circ}F}\left(1-\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta}}{\prod_{\zeta=1}^{{}^{\circ}F}\left(1+(\lambda-1)\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta} + (\lambda-1)\prod_{\zeta=1}^{{}^{\circ}F}\left(1-\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta}} = 1 - \frac{\lambda\prod_{\zeta=1}^{{}^{\circ}F}\left(1-\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta}}{\prod_{\zeta=1}^{{}^{\circ}F}\left(1+(\lambda-1)\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta} + (\lambda-1)\prod_{\zeta=1}^{{}^{\circ}F}\left(1-\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta}}$$

$$\le 1 - \frac{\lambda\prod_{\zeta=1}^{{}^{\circ}F}\left(1-\tilde{\chi}_{\tilde{\chi}_{\overline{\mho}_\zeta}}\right)^{\underline{\omega}_\zeta}}{\lambda}$$

$$= 1 - \prod_{\zeta=1}^{{}^{\circ}F}\left(1-\tilde{\chi}_{\tilde{\chi}_{\overline{\mho}_\zeta}}\right)^{\underline{\omega}_\zeta}.$$

Now for the complex part

$$\prod_{\zeta=1}^{{}^{\circ}F}\left(1+(\lambda-1)\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta} + (\lambda-1)\prod_{\zeta=1}^{{}^{\circ}F}\left(1-\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta} \le \sum_{\zeta=1}^{{}^{\circ}F}\underline{\omega}_\zeta\left(1+(\lambda-1)\tilde{Z}_{\overline{\mho}_\zeta}\right) + (\lambda-1)\sum_{\zeta=1}^{{}^{\circ}F}\underline{\omega}_\zeta\left(1+\tilde{Z}_{\overline{\mho}_\zeta}\right)$$

Then

$$\frac{\prod_{\zeta=1}^{{}^{\circ}F}\left(1+(\lambda-1)\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta} - \prod_{\zeta=1}^{{}^{\circ}F}\left(1-\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta}}{\prod_{\zeta=1}^{{}^{\circ}F}\left(1+(\lambda-1)\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta} + (\lambda-1)\prod_{\zeta=1}^{{}^{\circ}F}\left(1-\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta}} = 1 - \frac{\lambda\prod_{\zeta=1}^{{}^{\circ}F}\left(1-\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta}}{\prod_{\zeta=1}^{{}^{\circ}F}\left(1+(\lambda-1)\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta} + (\lambda-1)\prod_{\zeta=1}^{{}^{\circ}F}\left(1-\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta}}$$

$$\le 1 - \frac{\lambda\prod_{\zeta=1}^{{}^{\circ}F}\left(1-\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta}}{\lambda} = 1 - \prod_{\zeta=1}^{{}^{\circ}F}\left(1-\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta}$$

$$\Rightarrow \underset{\zeta=1}{\overset{{}^{\circ}F}{\oplus_H}}\underline{\omega}_\zeta\overline{\mho}_\zeta \le \underset{\zeta=1}{\overset{{}^{\circ}F}{\oplus}}\underline{\omega}_\zeta\overline{\mho}_\zeta$$

That is

$$CHF\mathrm{HW}A_{\lambda}(\overline{\mho}_1,\overline{\mho}_2,\dots,\overline{\mho}_{°F}) \le CHFHA(\overline{\mho}_1,\overline{\mho}_2,\dots,\overline{\mho}_{°F}).$$

Proposition 3. Let us suppose $\pi = (\overline{\mho}_1,\overline{\mho}_2,\dots,\overline{\mho}_{°F})$, be a set of °F $CHFEs$, if all $\overline{\mho}_\zeta(\zeta = 1,2,..,°F)$ are equal, i.e., $\overline{\mho}_\zeta = \overline{\mho}$ then

$$CHF\mathrm{HW}A_{\lambda}(\overline{\mho}_1,\overline{\mho}_2,\dots,\overline{\mho}_{°F}) = \overline{\mho}.$$

Proof: Suppose $\overline{\mho}_\zeta = \overline{\mho}$, for all $\zeta = 1,2,..,°F$, then we have

$$CHF\mathrm{HW}A_{\lambda}(\overline{\mho}_1,\overline{\mho}_2,\dots,\overline{\mho}_{°F}) = CHF\mathrm{HW}A_{\lambda}(\overline{\mho},\overline{\mho},\dots,\overline{\mho})$$

$$= \bigcup_{\tilde{\chi}_{\overline{\mho}}\epsilon\overline{\mho}}\left\{\left(\frac{\prod_{\zeta=1}^{°F}(1+(\lambda-1)\tilde{\chi}_{\overline{\mho}})^{\underline{\omega}_\zeta} - \prod_{\zeta=1}^{°F}(1-\tilde{\chi})^{\underline{\omega}_\zeta}}{\prod_{\zeta=1}^{°F}(1+(\lambda-1)\tilde{\chi}_{\overline{\mho}})^{\underline{\omega}_\zeta} + (\lambda-1)\prod_{\zeta=1}^{°F}(1-\tilde{\chi}_{\overline{\mho}})^{\underline{\omega}_\zeta}}\right)\right\}, \bigcup_{\tilde{Z}_{\overline{\mho}}\epsilon\overline{\mho}}\left\{\left(\frac{\prod_{\zeta=1}^{°F}(1+(\lambda-1)\tilde{Z}_{\overline{\mho}})^{\underline{\omega}_\zeta} - \prod_{\zeta=1}^{°F}(1-\tilde{Z}_{\overline{\mho}})^{\underline{\omega}_\zeta}}{\prod_{\zeta=1}^{°F}(1+(\lambda-1)\tilde{Z}_{\overline{\mho}})^{\underline{\omega}_\zeta} + (\lambda-1)\prod_{\zeta=1}^{°F}(1-\tilde{Z}_{\overline{\mho}})^{\underline{\omega}_\zeta}}\right)\right\}$$

$$= \bigcup_{\tilde{\chi}_{\overline{\mho}}\epsilon\overline{\mho}}\left\{\left(\frac{(1+(\lambda-1)\tilde{\chi}_{\overline{\mho}})\sum_{\zeta=1}^{°F}\underline{\omega}_\zeta - (1-\tilde{\chi}_{\overline{\mho}})\sum_{\zeta=1}^{°F}\underline{\omega}_\zeta}{(1+(\lambda-1)\tilde{\chi}_{\overline{\mho}})\sum_{\zeta=1}^{°F}\underline{\omega}_\zeta + (\lambda-1)(1-\tilde{\chi}_{\overline{\mho}})\sum_{\zeta=1}^{°F}\underline{\omega}_\zeta}\right)\right\}, \bigcup_{\tilde{Z}_{\overline{\mho}}\epsilon\overline{\mho}}\left\{\left(\frac{(1+(\lambda-1)\tilde{Z}_{\overline{\mho}})\sum_{\zeta=1}^{°F}\underline{\omega}_\zeta - (1-\tilde{Z}_{\overline{\mho}})\sum_{\zeta=1}^{°F}\underline{\omega}_\zeta}{(1+(\lambda-1)\tilde{Z}_{\overline{\mho}})\sum_{\zeta=1}^{°F}\underline{\omega}_\zeta + (\lambda-1)(1-\tilde{Z}_{\overline{\mho}})\sum_{\zeta=1}^{°F}\underline{\omega}_\zeta}\right)\right\}$$

$$= \bigcup_{\tilde{\chi}_{\overline{\mho}},\tilde{Z}_{\overline{\mho}}\epsilon\overline{\mho}}\{\mho\} = \mho.$$

Definition 8: Let us suppose $\pi = (\overline{\mho}_1,\overline{\mho}_2,\dots,\overline{\mho}_{°F})$, be the set of °F $CHFEs$. A CHFHWG operator is a function $C^{°F} \longrightarrow C$ Such that

$$CHF\mathrm{HW}G_{\lambda}(\overline{\mho}_1,\overline{\mho}_2,\dots,\overline{\mho}_{°F}) = \overline{\mho}_1^{\,\omega_1}\otimes_H\overline{\mho}_2^{\,\omega_2}\otimes_H \dots .\otimes_H\overline{\mho}_{°F}^{\,\underline{\omega}_{°F}} = \underset{\zeta=1}{\overset{°F}{\otimes_H}}\ \overline{\mho}_\zeta^{\,\underline{\omega}_\zeta},$$

Where the parameter λ must be a positive number and the $\underline{\omega}$ is weight vector of $\overline{\mho}_\zeta(\zeta = 1,2\dots,°F)$, is $\underline{\omega} = \left(\underline{\omega}_1,\underline{\omega}_2,\dots,\underline{\omega}_{°F}\right)^{T}$.Whose each weight for every argument belongs to unit interval and $\left(\sum_{\imath=1}^{°F}\underline{\omega}_\imath = 1\right)$.

Theorem 2. Let $\pi = (\overline{\mho}_1,\overline{\mho}_2,\dots,\overline{\mho}_{°F})$, consist of $CHFEs$, nonetheless the aggregated outcome by employing $CHF\mathrm{HW}G$ operator is a $CHFN$, and

$$CHF\mathrm{HW}G_{\lambda}(\overline{\mho}_1,\overline{\mho}_2,\dots,\overline{\mho}_{°F}) = \bigcup_{\tilde{\chi}_{\overline{\mho}_1},\tilde{Z}_{\overline{\mho}_1}\epsilon\overline{\mho}_1,\dots,\tilde{\chi}_{\overline{\mho}_{°F}},\tilde{Z}_{\overline{\mho}_{°F}}\epsilon\overline{\mho}_{°F}}\left\{\begin{array}{l}\left(\dfrac{\lambda\prod_{\zeta=1}^{°F}\left(\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta}}{\prod_{\zeta=1}^{°F}\left(1+(\lambda-1)\left(1-\tilde{\chi}_{\overline{\mho}_\zeta}\right)\right)^{\underline{\omega}_\zeta}+(\lambda-1)\prod_{\zeta=1}^{°F}\left(\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta}}\right)\\ +i\left(\dfrac{\lambda\prod_{\zeta=1}^{°F}\left(\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta}}{\prod_{\zeta=1}^{°F}\left(1+(\lambda-1)\left(1-\tilde{Z}_{\overline{\mho}_\zeta}\right)\right)^{\underline{\omega}_\zeta}+(\lambda-1)\prod_{\zeta=1}^{°F}\left(\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta}}\right)\end{array}\right\} \quad (2)$$

Proof: The proof is straightforward.

Remark 2. By considering $\lambda = 1$, then the $CHF\mathrm{HW}G$ operator will be converted in the shape, given below

$$CHFWG_{\lambda}(\overline{\mho}_1,\overline{\mho}_2,\dots,\overline{\mho}_{°F}) = \underset{\zeta=1}{\overset{°F}{\otimes_H}}\ \overline{\mho}_\zeta^{\,\underline{\omega}_\zeta} = \bigcup_{\tilde{\chi}_{\overline{\mho}_1},\tilde{Z}_{\overline{\mho}_1}\epsilon\overline{\mho}_1..,\tilde{\chi}_{\overline{\mho}_{°F}},\tilde{Z}_{\overline{\mho}_{°F}}\epsilon\overline{\mho}_{°F}}\left\{\prod_{\zeta=1}^{°F}\left(\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta} + i\prod_{\zeta=1}^{°F}\left(\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta}\right\}$$

Which is named as $CHFWG$ operator. If $\lambda = 2$, then the $CHF\mathrm{HW}G$ operator will be converted in the shape, given below

$$CHFWG_{\gimel}(\overline{\mho}_1,\overline{\mho}_2,\dots,\overline{\mho}_{{}^{\circ}\mathrm{F}}) = \underset{\zeta=1}{\overset{{}^{\circ}\mathrm{F}}{\otimes_{\varepsilon}}}\ \overline{\mho}_{\zeta}{}^{\underline{\omega}_{\zeta}}$$

$$= \bigcup_{\tilde{\chi}_{\overline{\mho}_1},\tilde{Z}_{\overline{\mho}_1}\epsilon\overline{\mho}_1\dots,\tilde{\chi}_{\overline{\mho}_{{}^{\circ}\mathrm{F}}},\tilde{Z}_{\overline{\mho}_{{}^{\circ}\mathrm{F}}}\epsilon\overline{\mho}_{{}^{\circ}\mathrm{F}}}\left\{\left(\frac{2\prod_{\zeta=1}^{{}^{\circ}\mathrm{F}}\left(\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta}}{\prod_{\zeta=1}^{{}^{\circ}\mathrm{F}}\left(2-\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta}+\prod_{\zeta=1}^{{}^{\circ}\mathrm{F}}\left(\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta}}\right)+i\left(\frac{2\prod_{\zeta=1}^{{}^{\circ}\mathrm{F}}\left(\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta}}{\prod_{\zeta=1}^{{}^{\circ}\mathrm{F}}\left(2-\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta}+\prod_{\zeta=1}^{{}^{\circ}\mathrm{F}}\left(\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta}}\right)\right\},$$

Which is named as CHF Einstein weighted geometric $(CHFEWG)$. Particularly, if $\underline{\omega} = \left(\frac{1}{{}^{\circ}\mathrm{F}},\frac{1}{{}^{\circ}\mathrm{F}},\dots,\frac{1}{{}^{\circ}\mathrm{F}}\right)^{\mathcal{T}}$, then the $CHF\mathrm{H}WG$ operator will be converted in the shape, given below

$$CHFHA(\overline{\mho}_1,\overline{\mho}_2,\dots,\overline{\mho}_{{}^{\circ}\mathrm{F}})$$

$$= \bigcup_{\tilde{\chi}_{\overline{\mho}_1},\tilde{Z}_{\overline{\mho}_1}\epsilon\overline{\mho}_1,\dots,\tilde{\chi}_{\overline{\mho}_{{}^{\circ}\mathrm{F}}},\tilde{Z}_{\overline{\mho}_{{}^{\circ}\mathrm{F}}}\epsilon\overline{\mho}_{{}^{\circ}\mathrm{F}}}\left\{\begin{array}{l}\left(\dfrac{\prod_{\zeta=1}^{{}^{\circ}\mathrm{F}}\left(1+(\gimel-1)\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\frac{1}{{}^{\circ}\mathrm{F}}}-\prod_{\zeta=1}^{{}^{\circ}\mathrm{F}}\left(1-\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\frac{1}{{}^{\circ}\mathrm{F}}}}{\prod_{\zeta=1}^{{}^{\circ}\mathrm{F}}\left(1+(\gimel-1)\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\frac{1}{{}^{\circ}\mathrm{F}}}+(\gimel-1)\prod_{\zeta=1}^{{}^{\circ}\mathrm{F}}\left(1-\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\frac{1}{{}^{\circ}\mathrm{F}}}}\right)\\ +i\left(\dfrac{\prod_{\zeta=1}^{{}^{\circ}\mathrm{F}}\left(1+(\gimel-1)\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\frac{1}{{}^{\circ}\mathrm{F}}}-\prod_{\zeta=1}^{{}^{\circ}\mathrm{F}}\left(1-\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\frac{1}{{}^{\circ}\mathrm{F}}}}{\prod_{\zeta=1}^{{}^{\circ}\mathrm{F}}\left(1+(\gimel-1)\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\frac{1}{{}^{\circ}\mathrm{F}}}+(\gimel-1)\prod_{\zeta=1}^{{}^{\circ}\mathrm{F}}\left(1-\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\frac{1}{{}^{\circ}\mathrm{F}}}}\right)\end{array}\right\},$$

Known as CHF Hamacher geometric $(CHFHG)$ operator.

Proposition 4. Let us suppose $\pi = (\overline{\mho}_1,\overline{\mho}_2,\dots,\overline{\mho}_{{}^{\circ}\mathrm{F}})$, be a set of ${}^{\circ}\mathrm{F}$ $CHFEs$ and the weight vector $\underline{\omega}$ of $\overline{\mho}_\zeta(\zeta = 1,2\dots,{}^{\circ}\mathrm{F})$ is $\underline{\omega} = \left(\underline{\omega}_1,\underline{\omega}_2,\dots,\underline{\omega}_{{}^{\circ}\mathrm{F}}\right)^{\mathcal{T}}$. Whose each weight belongs to unit interval and the sum of weights must be equal to one $\left(\sum_{\hat{\imath}=1}^{{}^{\circ}\mathrm{F}}\underline{\omega}_{\hat{\imath}} = 1\right)$.Then

1. $\underset{\zeta=1}{\overset{{}^{\circ}\mathrm{F}}{\oplus_H}}\ \underline{\omega}_\zeta\overline{\mho}_\zeta{}^{\dot{c}} = \left(\underset{\zeta=1}{\overset{{}^{\circ}\mathrm{F}}{\otimes_H}}\ \overline{\mho}_\zeta{}^{\underline{\omega}_\zeta}\right)^{\dot{c}}$
2. $\underset{\zeta=1}{\overset{{}^{\circ}\mathrm{F}}{\otimes_H}}\ \left(\overline{\mho}_\zeta{}^{\dot{c}}\right)^{\underline{\omega}_\zeta} = \left(\underset{\zeta=1}{\overset{{}^{\circ}\mathrm{F}}{\oplus_H}}\ \underline{\omega}_\zeta\overline{\mho}_\zeta\right)^{\dot{c}}$

Proof:

1. $\underset{\zeta=1}{\overset{{}^{\circ}\mathrm{F}}{\oplus_H}}\ \underline{\omega}_\zeta\overline{\mho}_\zeta{}^{\dot{c}}$

$$= \bigcup_{\tilde{\chi}_{\overline{\mho}_1},\tilde{Z}_{\overline{\mho}_1}\epsilon\overline{\mho}_1,\dots,\tilde{\chi}_{\overline{\mho}_{{}^{\circ}\mathrm{F}}},\tilde{Z}_{\overline{\mho}_{{}^{\circ}\mathrm{F}}}\epsilon\overline{\mho}_{{}^{\circ}\mathrm{F}}}\left\{\begin{array}{l}\left(\dfrac{\prod_{\zeta=1}^{{}^{\circ}\mathrm{F}}\left(1+(\gimel-1)\left(1-\tilde{\chi}_{\overline{\mho}_\zeta}\right)\right)^{\underline{\omega}_\zeta}-\prod_{\zeta=1}^{{}^{\circ}\mathrm{F}}\left(\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta}}{\prod_{\zeta=1}^{{}^{\circ}\mathrm{F}}\left(1+(\gimel-1)\left(1-\tilde{\chi}_{\overline{\mho}_\zeta}\right)\right)^{\underline{\omega}_\zeta}+(\gimel-1)\prod_{\zeta=1}^{{}^{\circ}\mathrm{F}}\left(\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta}}\right)\\ +i\left(\dfrac{\prod_{\zeta=1}^{{}^{\circ}\mathrm{F}}\left(1+(\gimel-1)\left(1-\tilde{Z}_{\overline{\mho}_\zeta}\right)\right)^{\underline{\omega}_\zeta}-\prod_{\zeta=1}^{{}^{\circ}\mathrm{F}}\left(\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta}}{\prod_{\zeta=1}^{{}^{\circ}\mathrm{F}}\left(1+(\gimel-1)\left(1-\tilde{Z}_{\overline{\mho}_\zeta}\right)\right)^{\underline{\omega}_\zeta}+(\gimel-1)\prod_{\zeta=1}^{{}^{\circ}\mathrm{F}}\left(\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta}}\right)\end{array}\right\}$$

$$= \bigcup_{\tilde{\chi}_{\overline{\mho}_1},\tilde{Z}_{\overline{\mho}_1}\epsilon\overline{\mho}_1,\dots,\tilde{\chi}_{\overline{\mho}_{{}^{\circ}\mathrm{F}}},\tilde{Z}_{\overline{\mho}_{{}^{\circ}\mathrm{F}}}\epsilon\overline{\mho}_{{}^{\circ}\mathrm{F}}}\left\{\begin{array}{l}\left(1-\dfrac{\gimel\prod_{\zeta=1}^{{}^{\circ}\mathrm{F}}\left(\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta}}{\prod_{\zeta=1}^{{}^{\circ}\mathrm{F}}\left(1+(\gimel-1)\left(1-\tilde{\chi}_{\overline{\mho}_\zeta}\right)\right)^{\underline{\omega}_\zeta}+(\gimel-1)\prod_{\zeta=1}^{{}^{\circ}\mathrm{F}}\left(\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta}}\right)\\ +i\left(1-\dfrac{\gimel\prod_{\zeta=1}^{{}^{\circ}\mathrm{F}}\left(\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta}}{\prod_{\zeta=1}^{{}^{\circ}\mathrm{F}}\left(1+(\gimel-1)\left(1-\tilde{Z}_{\overline{\mho}_\zeta}\right)\right)^{\underline{\omega}_\zeta}+(\gimel-1)\prod_{\zeta=1}^{{}^{\circ}\mathrm{F}}\left(\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta}}\right)\end{array}\right\}$$

$$=\left(\bigotimes_{H\,\zeta=1}^{\circ F}\overline{\mho}_{\zeta}^{\,\underline{\omega}_{\zeta}}\right)^{\dot{c}}$$

2.

$$\bigotimes_{H\,\zeta=1}^{\circ F}\left(\overline{\mho}_{\zeta}^{\,\dot{c}}\right)^{\underline{\omega}_{\zeta}}$$

$$=\bigcup_{\tilde{\chi}_{\overline{\mho}_1},\tilde{Z}_{\overline{\mho}_1}\epsilon\overline{\mho}_1,\dots,\tilde{\chi}_{\overline{\mho}_{\circ F}},\tilde{Z}_{\overline{\mho}_{\circ F}}\epsilon\overline{\mho}_{\circ F}}\left\{\begin{array}{c}\left(\dfrac{\lambda\prod_{\zeta=1}^{\circ F}\left(1-\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta}}{\prod_{\zeta=1}^{\circ F}\left(1+(\lambda-1)\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta}+(\lambda-1)\prod_{\zeta=1}^{\circ F}\left(1-\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta}}\right)\\+\imath\left(\dfrac{\lambda\prod_{\zeta=1}^{\circ F}\left(1-\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta}}{\prod_{\zeta=1}^{\circ F}\left(1+(\lambda-1)\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta}+(\lambda-1)\prod_{\zeta=1}^{\circ F}\left(1-\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta}}\right)\end{array}\right\}$$

$$=\bigcup_{\tilde{\chi}_{\overline{\mho}_1},\tilde{Z}_{\overline{\mho}_1}\epsilon\overline{\mho}_1,\dots,\tilde{\chi}_{\overline{\mho}_{\circ F}},\tilde{Z}_{\overline{\mho}_{\circ F}}\epsilon\overline{\mho}_{\circ F}}\left\{\begin{array}{c}\left(1-\dfrac{\prod_{\zeta=1}^{\circ F}\left(1+(\lambda-1)\left(\tilde{\chi}_{\overline{\mho}_\zeta}\right)\right)^{\underline{\omega}_\zeta}-\prod_{\zeta=1}^{\circ F}\left(1-\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta}}{\prod_{\zeta=1}^{\circ F}\left(1+(\lambda-1)\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta}+(\lambda-1)\prod_{\zeta=1}^{\circ F}\left(1-\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta}}\right)\\+\imath\left(1-\dfrac{\prod_{\zeta=1}^{\circ F}\left(1+(\lambda-1)\left(\tilde{Z}_{\overline{\mho}_\zeta}\right)\right)^{\underline{\omega}_\zeta}-\prod_{\zeta=1}^{\circ F}\left(1-\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta}}{\prod_{\zeta=1}^{\circ F}\left(1+(\lambda-1)\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta}+(\lambda-1)\prod_{\zeta=1}^{\circ F}\left(1-\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\underline{\omega}_\zeta}}\right)\end{array}\right\}$$

$$=\left(\bigoplus_{H\,\zeta=1}^{\circ F}\underline{\omega}_\zeta\overline{\mho}_\zeta\right)^{\dot{c}}.$$

Proposition 5. Let us suppose $\pi=(\overline{\mho}_1,\overline{\mho}_2,\dots,\overline{\mho}_{\circ F})$, be a set of $\circ F$ $CHFEs$. $\overline{\mho}^-=\min\left\{\tilde{\chi}_{\overline{\mho}_\zeta}{}^-|\tilde{\chi}_{\overline{\mho}_\zeta}{}^-\epsilon\overline{\mho}_\zeta{}^-\right\}+\imath\min\left\{\tilde{Z}_{\overline{\mho}_\zeta}{}^-|\tilde{Z}_{\overline{\mho}_\zeta}{}^-\epsilon\overline{\mho}_\zeta{}^-\right\}$ and $\overline{\mho}^+=\min\left\{\tilde{\chi}_{\overline{\mho}_\zeta}{}^+|\tilde{\chi}_{\overline{\mho}_\zeta}{}^+\epsilon\overline{\mho}_\zeta{}^+\right\}+\imath\min\left\{\tilde{Z}_{\overline{\mho}_\zeta}{}^+|\tilde{Z}_{\overline{\mho}_\zeta}{}^+\epsilon\overline{\mho}_\zeta{}^+\right\}$ then

$$\overline{\mho}^-\leq HF\mathrm{H}WA_\lambda(\overline{\mho}_1,\overline{\mho}_2,\dots,\overline{\mho}_{\circ F})\leq\overline{\mho}^+.$$

Proof: Let $\mathfrak{f}(\underline{\delta})=\frac{1+(\lambda-1)\underline{\delta}}{1-\underline{\delta}}$, $\underline{\delta}\epsilon[0,1)$, then $\mathfrak{f}(\underline{\delta})=\left(\frac{1+(\lambda-1)\underline{\delta}}{1-\underline{\delta}}\right)'=\frac{\lambda}{(1-\partial)^2}>0$, thus $\mathfrak{f}(\underline{\delta})$ is an ascending function. Since for all ζ, $\tilde{\chi}_{\overline{\mho}_\zeta}{}^-+\tilde{Z}_{\overline{\mho}_\zeta}{}^-\leq\tilde{\chi}_{\overline{\mho}_\zeta}+\tilde{Z}_{\overline{\mho}_\zeta}\leq\tilde{\chi}_{\overline{\mho}_\zeta}{}^++\tilde{Z}_{\overline{\mho}_\zeta}{}^+\Rightarrow\mathfrak{f}\left(\tilde{\chi}_{\overline{\mho}_\zeta}{}^-+\tilde{Z}_{\overline{\mho}_\zeta}{}^-\right)\leq\mathfrak{f}\left(\tilde{\chi}_{\overline{\mho}_\zeta}+\tilde{Z}_{\overline{\mho}_\zeta}\right)\leq\mathfrak{f}\left(\tilde{\chi}_{\overline{\mho}_\zeta}{}^++\tilde{Z}_{\overline{\mho}_\zeta}{}^+\right)$. For this, we have to prove that, $\tilde{\chi}_{\overline{\mho}_\zeta}{}^-\leq\tilde{\chi}_{\overline{\mho}_\zeta}\tilde{Z}_{\overline{\mho}_\zeta}{}^-\leq\tilde{Z}_{\overline{\mho}_\zeta}and\tilde{\chi}_{\overline{\mho}_\zeta}\leq\tilde{\chi}_{\overline{\mho}_\zeta}{}^+,\tilde{Z}_{\overline{\mho}_\zeta}\leq\tilde{Z}_{\overline{\mho}_\zeta}{}^+$

$$\frac{1+(\lambda-1)\,\tilde{\chi}_{\overline{\mho}_\zeta}{}^-}{1-\,\tilde{\chi}_{\overline{\mho}_\zeta}{}^-}\leq\frac{1+(\lambda-1)\tilde{\chi}_{\overline{\mho}_\zeta}}{1-\tilde{\chi}_{\overline{\mho}_\zeta}}\text{ and }\frac{1+(\lambda-1)\tilde{Z}_{\overline{\mho}_\zeta}{}^-}{1-\tilde{Z}_{\overline{\mho}_\zeta}{}^-}\leq\frac{1+(\lambda-1)\tilde{Z}_{\overline{\mho}_\zeta}}{1-\tilde{Z}_{\overline{\mho}_\zeta}}.$$

Now we have

$$\left(\frac{1+(\lambda-1)\,\tilde{\chi}_{\overline{\mho}_\zeta}{}^-}{1-\,\tilde{\chi}_{\overline{\mho}_\zeta}{}^-}\right)^{\underline{\omega}_\zeta}\leq\left(\frac{1+(\lambda-1)\tilde{\chi}_{\overline{\mho}_\zeta}}{1-\tilde{\chi}_{\overline{\mho}_\zeta}}\right)^{\underline{\omega}_\zeta}\text{ and }\left(\frac{1+(\lambda-1)\,\tilde{Z}_{\overline{\mho}_\zeta}{}^-}{1-\,\tilde{Z}_{\overline{\mho}_\zeta}{}^-}\right)^{\underline{\omega}_\zeta}\leq\left(\frac{1+(\lambda-1)\tilde{Z}_{\overline{\mho}_\zeta}}{1-\tilde{Z}_{\overline{\mho}_\zeta}}\right)^{\underline{\omega}_\zeta}$$

$$\Leftrightarrow\left(1+\frac{\lambda\,\tilde{\chi}_{\overline{\mho}_\zeta}{}^-}{1-\,\tilde{\chi}_{\overline{\mho}_\zeta}{}^-}\right)^{\underline{\omega}_\zeta}\leq\left(\frac{1+(\lambda-1)\tilde{\chi}_{\overline{\mho}_\zeta}}{1-\tilde{\chi}_{\overline{\mho}_\zeta}}\right)^{\underline{\omega}_\zeta}\text{ and }\left(1+\frac{\lambda\,\tilde{Z}_{\overline{\mho}_\zeta}{}^-}{1-\,\tilde{Z}_{\overline{\mho}_\zeta}{}^-}\right)^{\underline{\omega}_\zeta}\leq\left(\frac{1+(\lambda-1)\tilde{Z}_{\overline{\mho}_\zeta}}{1-\tilde{Z}_{\overline{\mho}_\zeta}}\right)^{\underline{\omega}_\zeta}$$

$$\Leftrightarrow\prod_{\zeta=1}^{\circ F}\left(1+\frac{\lambda\,\tilde{\chi}_{\overline{\mho}_\zeta}{}^-}{1-\,\tilde{\chi}_{\overline{\mho}_\zeta}{}^-}\right)^{\underline{\omega}_\zeta}\leq\prod_{\zeta=1}^{\circ F}\left(\frac{1+(\lambda-1)\tilde{\chi}_{\overline{\mho}_\zeta}}{1-\tilde{\chi}_{\overline{\mho}_\zeta}}\right)^{\underline{\omega}_\zeta}\text{ and }\prod_{\zeta=1}^{\circ F}\left(1+\frac{\lambda\,\tilde{Z}_{\overline{\mho}_\zeta}{}^-}{1-\,\tilde{Z}_{\overline{\mho}_\zeta}{}^-}\right)^{\underline{\omega}_\zeta}\leq\prod_{\zeta=1}^{\circ F}\left(\frac{1+(\lambda-1)\tilde{Z}_{\overline{\mho}_\zeta}}{1-\tilde{Z}_{\overline{\mho}_\zeta}}\right)^{\underline{\omega}_\zeta}$$

Using lemma 1, where $\sum_{\zeta=1}^{\circ F}\underline{\omega}_\zeta=1$, so

$$\Leftrightarrow \left(1+\frac{\lambda\,\tilde{\chi}_{\overline{\mho}_\zeta}{}^{-}}{1-\tilde{\chi}_{\overline{\mho}_\zeta}{}^{-}}\right)^{\omega_\zeta} \le \prod_{\zeta=1}^{\circ\mathrm{F}}\left(\frac{1+(\lambda-1)\tilde{\chi}_{\overline{\mho}_\zeta}}{1-\tilde{\chi}_{\overline{\mho}_\zeta}}\right)^{\omega_\zeta} \text{ and } \left(1+\frac{\lambda\,\tilde{Z}_{\overline{\mho}_\zeta}{}^{-}}{1-\tilde{Z}_{\overline{\mho}_\zeta}{}^{-}}\right)^{\omega_\zeta} \le \prod_{\zeta=1}^{\circ\mathrm{F}}\left(\frac{1+(\lambda-1)\tilde{Z}_{\overline{\mho}_\zeta}}{1-\tilde{Z}_{\overline{\mho}_\zeta}}\right)^{\omega_\zeta}$$

$$\Leftrightarrow \lambda+\frac{\lambda\,\tilde{\chi}_{\overline{\mho}_\zeta}{}^{-}}{1-\tilde{\chi}_{\overline{\mho}_\zeta}{}^{-}} \le \prod_{\zeta=1}^{\circ\mathrm{F}}\left(\frac{1+(\lambda-1)\tilde{\chi}_{\overline{\mho}_\zeta}}{1-\tilde{\chi}_{\overline{\mho}_\zeta}}\right)^{\omega_\zeta}+(\lambda-1) \text{and } \lambda+\frac{\lambda\,\tilde{Z}_{\overline{\mho}_\zeta}{}^{-}}{1-\tilde{Z}_{\overline{\mho}_\zeta}{}^{-}} \le \prod_{\zeta=1}^{\circ\mathrm{F}}\left(\frac{1+(\lambda-1)\tilde{Z}_{\overline{\mho}_\zeta}}{1-\tilde{Z}_{\overline{\mho}_\zeta}}\right)^{\omega_\zeta}+(\lambda-1)$$

$$\Leftrightarrow \frac{1}{\lambda+\frac{\lambda\,\tilde{\chi}_{\overline{\mho}_\zeta}{}^{-}}{1-\tilde{\chi}_{\overline{\mho}_\zeta}{}^{-}}} \le \frac{1}{\prod_{\zeta=1}^{\circ\mathrm{F}}\left(\frac{1+(\lambda-1)\tilde{\chi}_{\overline{\mho}_\zeta}}{1-\tilde{\chi}_{\overline{\mho}_\zeta}}\right)^{\omega_\zeta}+(\lambda-1)} \text{ and } \frac{1}{\lambda+\frac{\lambda\,\tilde{Z}_{\overline{\mho}_\zeta}{}^{-}}{1-\tilde{Z}_{\overline{\mho}_\zeta}{}^{-}}} \le \frac{1}{\prod_{\zeta=1}^{\circ\mathrm{F}}\left(\frac{1+(\lambda-1)\tilde{Z}_{\overline{\mho}_\zeta}}{1-\tilde{Z}_{\overline{\mho}_\zeta}}\right)^{\omega_\zeta}+(\lambda-1)}$$

$$\Leftrightarrow \frac{1-\tilde{\chi}_{\overline{\mho}_\zeta}{}^{-}}{\lambda} \le \frac{\prod_{\zeta=1}^{\circ\mathrm{F}}\left(1-\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\omega_\zeta}}{\prod_{\zeta=1}^{\circ\mathrm{F}}\left(1+(\lambda-1)\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\omega_\zeta}+(\lambda-1)\prod_{\zeta=1}^{\circ\mathrm{F}}\left(1-\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\omega_\zeta}} \text{ and } \frac{1-\tilde{Z}_{\overline{\mho}_\zeta}{}^{-}}{\lambda} \le \frac{\prod_{\zeta=1}^{\circ\mathrm{F}}\left(1-\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\omega_\zeta}}{\prod_{\zeta=1}^{\circ\mathrm{F}}\left(1+(\lambda-1)\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\omega_\zeta}+(\lambda-1)\prod_{\zeta=1}^{\circ\mathrm{F}}\left(1-\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\omega_\zeta}}$$

$$\Leftrightarrow 1-\tilde{\chi}_{\overline{\mho}_\zeta}{}^{-} \le \frac{\lambda\prod_{\zeta=1}^{\circ\mathrm{F}}\left(1-\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\omega_\zeta}}{\prod_{\zeta=1}^{\circ\mathrm{F}}\left(1+(\lambda-1)\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\omega_\zeta}+(\lambda-1)\prod_{\zeta=1}^{\circ\mathrm{F}}\left(1-\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\omega_\zeta}} \text{ and } 1-\tilde{Z}_{\overline{\mho}_\zeta}{}^{-} \le \frac{\lambda\prod_{\zeta=1}^{\circ\mathrm{F}}\left(1-\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\omega_\zeta}}{\prod_{\zeta=1}^{\circ\mathrm{F}}\left(1+(\lambda-1)\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\omega_\zeta}+(\lambda-1)\prod_{\zeta=1}^{\circ\mathrm{F}}\left(1-\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\omega_\zeta}}$$

$$\Leftrightarrow \tilde{\chi}_{\overline{\mho}_\zeta}{}^{-} \le 1-\frac{\lambda\prod_{\zeta=1}^{\circ\mathrm{F}}\left(1-\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\omega_\zeta}}{\prod_{\zeta=1}^{\circ\mathrm{F}}\left(1+(\lambda-1)\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\omega_\zeta}+(\lambda-1)\prod_{\zeta=1}^{\circ\mathrm{F}}\left(1-\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\omega_\zeta}} \text{ and } \tilde{Z}_{\overline{\mho}_\zeta}{}^{-} \le 1-\frac{\lambda\prod_{\zeta=1}^{\circ\mathrm{F}}\left(1-\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\omega_\zeta}}{\prod_{\zeta=1}^{\circ\mathrm{F}}\left(1+(\lambda-1)\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\omega_\zeta}+(\lambda-1)\prod_{\zeta=1}^{\circ\mathrm{F}}\left(1-\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\omega_\zeta}}$$

$$\Leftrightarrow \tilde{\chi}_{\overline{\mho}_\zeta}{}^{-} \le \frac{\prod_{\zeta=1}^{\circ\mathrm{F}}\left(1+(\lambda-1)\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\omega_\zeta}-\prod_{\zeta=1}^{\circ\mathrm{F}}\left(1-\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\omega_\zeta}}{\prod_{\zeta=1}^{\circ\mathrm{F}}\left(1+(\lambda-1)\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\omega_\zeta}+(\lambda-1)\prod_{\zeta=1}^{\circ\mathrm{F}}\left(1-\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\omega_\zeta}} \text{ and } \tilde{Z}_{\overline{\mho}_\zeta}{}^{-} \le \frac{\prod_{\zeta=1}^{\circ\mathrm{F}}\left(1+(\lambda-1)\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\omega_\zeta}-\prod_{\zeta=1}^{\circ\mathrm{F}}\left(1-\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\omega_\zeta}}{\prod_{\zeta=1}^{\circ\mathrm{F}}\left(1+(\lambda-1)\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\omega_\zeta}+(\lambda-1)\prod_{\zeta=1}^{\circ\mathrm{F}}\left(1-\tilde{Z}_{\overline{\mho}_\zeta}\right)^{\omega_\zeta}}$$

We prove that $\tilde{\chi}_{\overline{\mho}_\zeta}{}^{-} \le \tilde{\chi}_{\overline{\mho}_\zeta}$, $\tilde{Z}_{\overline{\mho}_\zeta}{}^{-} \le \tilde{Z}_{\overline{\mho}_\zeta}$

$$\Rightarrow \overline{\mho}^{-} \le HFHWA_\lambda(\overline{\mho}_1, \overline{\mho}_2, \dots, \overline{\mho}_{\circ\mathrm{F}}) \quad (I).$$

We can easily prove $\tilde{\chi}_{\overline{\mho}_\zeta} \le \tilde{\chi}_{\overline{\mho}_\zeta}{}^{+}$, $\tilde{Z}_{\overline{\mho}_\zeta} \le \tilde{Z}_{\overline{\mho}_\zeta}{}^{+}$

$$\Rightarrow HFHWA_\lambda(\overline{\mho}_1, \overline{\mho}_2, \dots, \overline{\mho}_{\circ\mathrm{F}}) \le \overline{\mho}^{+} \quad (II).$$

From (I) and (II), we can write,

$$\overline{\mho}^{-} \le HFHWA_\lambda(\overline{\mho}_1, \overline{\mho}_2, \dots, \overline{\mho}_{\circ\mathrm{F}}) \le \overline{\mho}^{+}.$$

Definition 9. Let us suppose $\pi = (\overline{\mho}_1, \overline{\mho}_2, \dots, \overline{\mho}_{\circ\mathrm{F}})$, be a set of $\circ\mathrm{F}$ $CHFEs$. A GCHFHWA operator is a mapping $C^{\circ\mathrm{F}} \longrightarrow C$, such that

$$GCHFHWA_{\lambda,7}(\overline{\mho}_1, \overline{\mho}_2, \dots, \overline{\mho}_{\circ\mathrm{F}}) = \omega_1\overline{\mho}_1{}^{7} \oplus_H \omega_2\overline{\mho}_2{}^{7} \oplus_H \dots . \oplus_H \underline{\omega}_{\circ\mathrm{F}}\overline{\mho}_{\circ\mathrm{F}}{}^{7} = \left(\underset{\zeta=1}{\overset{\circ\mathrm{F}}{\oplus_H}}\ \underline{\omega}_\zeta\overline{\mho}_\zeta{}^{7}\right)^{\frac{1}{7}},$$

Where the parameter λ, must be a positive number and the $\underline{\omega} = \left(\underline{\omega}_1, \underline{\omega}_2, \dots, \underline{\omega}_{\circ\mathrm{F}}\right)^{T}$, is the weight vector of $\overline{\mho}_\zeta(\zeta = 1, 2 \dots, \circ\mathrm{F})$,whose weight for every argument belongs to the unit interval and $\sum_{\mathrm{i}=1}^{\circ\mathrm{F}} \underline{\omega}_{\mathrm{i}} = 1$.

Theorem 3. Let $\pi = (\overline{\mho}_1, \overline{\mho}_2, \dots, \overline{\mho}_{\circ\mathrm{F}})$, consist of $CHFEs$, nonetheless the aggregated outcome by employing $GCHFHWA$ operator is a $CHFN$, and

$GCHFHWA_{\lambda,7}(\overline{\mho}_1, \overline{\mho}_2, \dots, \overline{\mho}_{\circ\mathrm{F}}) =$

$$\bigcup_{\tilde{\chi}_{\overline{\mho}_1},\tilde{z}_{\overline{\mho}_1}\in\overline{\mho}_1,\ldots,\tilde{\chi}_{\overline{\mho}_{°F}},\tilde{z}_{\overline{\mho}_{°F}}\in\overline{\mho}_{°F}}\left\{\left(\frac{\prod_{\zeta=1}^{°F}\left((1+(\lambda-1))\frac{\lambda\left(\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\daleth}}{\left(1+(\lambda-1)\left(1-\tilde{\chi}_{\overline{\mho}_\zeta}\right)\right)^{\daleth}+(\lambda-1)\left(\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\daleth}}\right)^{\underline{\omega}\zeta}-\prod_{\zeta=1}^{°F}\left(1-\frac{\lambda\left(\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\daleth}}{\left(1+(\lambda-1)\left(1-\tilde{\chi}_{\overline{\mho}_\zeta}\right)\right)^{\daleth}+(\lambda-1)\left(\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\daleth}}\right)^{\underline{\omega}\zeta}}{\prod_{\zeta=1}^{°F}\left((1+(\lambda-1))\frac{\lambda\left(\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\daleth}}{\left(1+(\lambda-1)\left(1-\tilde{\chi}_{\overline{\mho}_\zeta}\right)\right)^{\daleth}+(\lambda-1)\left(\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\daleth}}\right)^{\underline{\omega}\zeta}+(\lambda-1)\prod_{\zeta=1}^{°F}\left(1-\frac{\lambda\left(\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\daleth}}{\left(1+(\lambda-1)\left(1-\tilde{\chi}_{\overline{\mho}_\zeta}\right)\right)^{\daleth}+(\lambda-1)\left(\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\daleth}}\right)^{\underline{\omega}\zeta}}\right)^{\frac{1}{\daleth}}+i\left(\frac{\prod_{\zeta=1}^{°F}\left((1+(\lambda-1))\frac{\lambda\left(\tilde{z}_{\overline{\mho}_\zeta}\right)^{\daleth}}{\left(1+(\lambda-1)\left(1-\tilde{z}_{\overline{\mho}_\zeta}\right)\right)^{\daleth}+(\lambda-1)\left(\tilde{z}_{\overline{\mho}_\zeta}\right)^{\daleth}}\right)^{\underline{\omega}\zeta}-\prod_{\zeta=1}^{°F}\left(1-\frac{\lambda\left(\tilde{z}_{\overline{\mho}_\zeta}\right)^{\daleth}}{\left(1+(\lambda-1)\left(1-\tilde{z}_{\overline{\mho}_\zeta}\right)\right)^{\daleth}+(\lambda-1)\left(\tilde{z}_{\overline{\mho}_\zeta}\right)^{\daleth}}\right)^{\underline{\omega}\zeta}}{\prod_{\zeta=1}^{°F}\left((1+(\lambda-1))\frac{\lambda\left(\tilde{z}_{\overline{\mho}_\zeta}\right)^{\daleth}}{\left(1+(\lambda-1)\left(1-\tilde{z}_{\overline{\mho}_\zeta}\right)\right)^{\daleth}+(\lambda-1)\left(\tilde{z}_{\overline{\mho}_\zeta}\right)^{\daleth}}\right)^{\underline{\omega}\zeta}+(\lambda-1)\prod_{\zeta=1}^{°F}\left(1-\frac{\lambda\left(\tilde{z}_{\overline{\mho}_\zeta}\right)^{\daleth}}{\left(1+(\lambda-1)\left(1-\tilde{z}_{\overline{\mho}_\zeta}\right)\right)^{\daleth}+(\lambda-1)\left(\tilde{z}_{\overline{\mho}_\zeta}\right)^{\daleth}}\right)^{\underline{\omega}\zeta}}\right)^{\frac{1}{\daleth}}\right\} \quad (3)$$

Proof: The proof is straightforward.

Definition 10. Let us suppose $\pi = (\overline{\mho}_1, \overline{\mho}_2, \ldots, \overline{\mho}_{°F})$, be a set of °F $CHFEs$. A GCHFHWG operator is a mapping $C^{°F} \longrightarrow C$, such that

$$GCHFHWG_{\lambda,\daleth}(\overline{\mho}_1, \overline{\mho}_2, \ldots, \overline{\mho}_{°F}) = \frac{1}{\daleth}\left((\daleth\overline{\mho}_1)^{\omega_1}\otimes_H(\daleth\overline{\mho}_2)^{\omega_2}\otimes_H \ldots . \otimes_H(\daleth\overline{\mho}_{°F})^{\underline{\omega}_{\ddot{n}}}\right) = \frac{1}{\daleth}\bigotimes_{H\,\zeta=1}^{°F}\left(\daleth\overline{\mho}_\zeta\right)^{\underline{\omega}\zeta},$$

Theorem 4. Let $\pi = (\overline{\mho}_1, \overline{\mho}_2, \ldots, \overline{\mho}_{°F})$, consist of $CHFEs$, nonetheless the aggregated outcome by employing $GCHFHWG$ operator is a $CHFN$, and,

$CGHFHWG_{\lambda,\daleth}(\overline{\mho}_1, \overline{\mho}_2, \ldots, \overline{\mho}_{°F}) =$

$$\bigcup_{\tilde{\chi}_{\overline{\mho}_1},\tilde{z}_{\overline{\mho}_1}\in\overline{\mho}_1,\ldots,\tilde{\chi}_{\overline{\mho}_{°F}},\tilde{z}_{\overline{\mho}_{°F}}\in\overline{\mho}_{°F}}\left\{\begin{array}{l}1-\left(1-\frac{\lambda\prod_{\zeta=1}^{°F}\left(\frac{\left(1+(\lambda-1)\left(1-\tilde{\chi}_{\overline{\mho}_\zeta}\right)\right)^{\daleth}-\left(1-\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\daleth}}{\left(1+(\lambda-1)\left(\tilde{\chi}_{\overline{\mho}_\zeta}\right)\right)^{\daleth}+(\lambda-1)\left(1-\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\daleth}}\right)^{\underline{\omega}\zeta}}{\left(\prod_{\zeta=1}^{°F}\left(\left(1-\frac{(1+(\lambda-1))\times\left(1+(\lambda-1)\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\daleth}-\left(1-\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\daleth}}{\left(1+(\lambda-1)\left(\tilde{\chi}_{\overline{\mho}_\zeta}\right)\right)^{\daleth}+(\lambda-1)\left(1-\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\daleth}}\right)\right)^{\underline{\omega}\zeta}+(\lambda-1)\prod_{\zeta=1}^{°F}\left(\frac{\left(1+(\lambda-1)\left(\tilde{\chi}_{\overline{\mho}_\zeta}\right)\right)^{\daleth}-\left(1-\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\daleth}}{\left(1+(\lambda-1)\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\daleth}+(\lambda-1)\left(1-\tilde{\chi}_{\overline{\mho}_\zeta}\right)^{\daleth}}\right)^{\underline{\omega}\zeta}\right)}\right)^{\frac{1}{\daleth}}+i\\ \left(1-\left(1-\frac{\lambda\prod_{\zeta=1}^{°F}\left(\frac{\left(1+(\lambda-1)\left(1-\tilde{z}_{\overline{\mho}_\zeta}\right)\right)^{\daleth}-\left(1-\tilde{z}_{\overline{\mho}_\zeta}\right)^{\daleth}}{\left(1+(\lambda-1)\left(\tilde{z}_{\overline{\mho}_\zeta}\right)\right)^{\daleth}+(\lambda-1)\left(1-\tilde{z}_{\overline{\mho}_\zeta}\right)^{\daleth}}\right)^{\underline{\omega}\zeta}}{\left(\prod_{\zeta=1}^{°F}\left((1+(\lambda-1))\left(1-\frac{\left(1+(\lambda-1)\tilde{z}_{\overline{\mho}_\zeta}\right)^{\daleth}-\left(1-\tilde{z}_{\overline{\mho}_\zeta}\right)^{\daleth}}{\left(1+(\lambda-1)\left(\tilde{z}_{\overline{\mho}_\zeta}\right)\right)^{\daleth}+(\lambda-1)\left(1-\tilde{z}_{\overline{\mho}_\zeta}\right)^{\daleth}}\right)\right)^{\underline{\omega}\zeta}+(\lambda-1)\prod_{\zeta=1}^{°F}\left(\frac{\left(1+(\lambda-1)\left(\tilde{z}_{\overline{\mho}_\zeta}\right)\right)^{\daleth}-\left(1-\tilde{z}_{\overline{\mho}_\zeta}\right)^{\daleth}}{\left(1+(\lambda-1)\tilde{z}_{\overline{\mho}_\zeta}\right)^{\daleth}+(\lambda-1)\left(1-\tilde{z}_{\overline{\mho}_\zeta}\right)^{\daleth}}\right)^{\underline{\omega}\zeta}\right)}\right)^{\frac{1}{\daleth}}\right)\end{array}\right\} \quad (4)$$

Proof: The proof is straightforward.

4. Applications of Hamacher Aggregation Operators Based on CHFS

Throughout this segment, we cover the MCDM technique and address numerical problems.

4.1 Technique for MCDM Based on Hamacher AOs

In this part, complex hesitant Hamacher AOs are applied to formulate an MCDM model where the uncertain information is hesitant fuzzy. The MCDM process entails selecting the best alternative from among many feasible options by ranking them according to a set of criteria or attributes. In this example, let $£ = (ꝑ_1, ꝑ_2, ., ., ꝑ_{ɰ})$, be the alternatives and $Ꝓ = (ʊ_1, ʊ_2, ., ., ʊ_{°F})$ be the criteria set with the criteria weighted vectors $\underline{ɷ} = \left(\underline{ɷ}_1, \underline{ɷ}_2, \dots, \underline{ɷ}_{°F}\right)^{T}$, where $\underline{ɷ}_\zeta$ must be positive and ζ is from 1to °F and sum of all weights must be equal to one. Suppose that the evaluated value of the alternative $ꝑ_{ɩ}(ɩ = 1, 2, \dots, ɰ)$ with criteria $ʊ_\zeta(\zeta = 1, 2 \dots, °F)$ is represented by a $CHFN\, \overline{Ʊ}_{ɩ\zeta} = \bigcup_{\tilde{\chi}_{\overline{Ʊ}_{ɩ\zeta}}, \tilde{Z}_{\overline{Ʊ}_{ɩ\zeta}} \epsilon \overline{Ʊ}_{ɩ\zeta}} \left\{\tilde{\chi}_{\overline{Ʊ}_{ɩ\zeta}} + i\tilde{Z}_{\overline{Ʊ}_{ɩ\zeta}}\right\} \epsilon \mathbb{H}$. This set contains the possible estimated values of the alternative $ꝑ_{ɩ}\epsilon £$ in terms of criteria $ʊ_\zeta \epsilon Ꝓ$. Consequently, a CHF decision matrix (CHFDM) $\tilde{\overline{\mathcal{D}}} = \left(\overline{Ʊ}_{ɩ\zeta}\right)_{ɰ\times °F}$, can be constructed.

In general, MCDM modeling without additional importance descriptions results in two types of criteria: cost-based, where the minimum attribute value is preferred, and benefit-based, where the maximum attribute value is preferred. If all standards $ʊ_\zeta(\zeta = 1, 2 \dots, °F)$ are of not different type, then the criteria values need not to be normalize. MCDM Problems have often been discussed separately in terms of cost and benefit attributes, but when issues involve both, they should not clash; if important, at least some unity can be achieved using Eq. (5) given below:

$$Ʋ_{ɩ\zeta} = \begin{cases} \overline{Ʊ}_{ɩ\zeta}, & \text{for benefit attribute } ʊ_\zeta \\ \left(\overline{Ʊ}_{ɩ\zeta}\right)^{\dot{c}}, & \text{for cost attribute } ʊ_\zeta \end{cases}, ɩ = 1, 2, \&, ɰ; \zeta = 1, 2\&, °F \quad (5)$$

The compliment of $\overline{Ʊ}_{ɩ\zeta}$, is denoted as

$$\left(\overline{Ʊ}_{ɩ\zeta}\right)^{\dot{c}} = \bigcup_{\tilde{\chi}_{\overline{Ʊ}_{ɩ\zeta}}, \tilde{Z}_{\overline{Ʊ}_{ɩ\zeta}} \epsilon \overline{Ʊ}_{ɩ\zeta}} \left\{\left(1 - \tilde{\chi}_{\overline{Ʊ}_{ɩ\zeta}}\right) + i\left(1 - \tilde{Z}_{\overline{Ʊ}_{ɩ\zeta}}\right)\right\},$$

in this result in a change in form making the matrix $\mathcal{H} = \left(Ʋ_{ɩ\zeta}\right)_{ɰ\times °F}$. To see how they can reach the optimal solution, the CHFHWA or CHFHWG operator was meant to be used for building up the hesitant fuzzy MCDM process, which contains the following steps:

i. The solution finder provides their alternatives according to the criteria.
ii. According to Eq.(5), convert the matrix $\tilde{\overline{\mathcal{D}}} = \left(\overline{Ʊ}_{ɩ\zeta}\right)_{ɰ\times °F}$ into $\mathcal{H} = \left(Ʋ_{ɩ\zeta}\right)_{ɰ\times °F}$
iii. Determine all the values of $Ʋ_{ɩ\zeta}$ $(\zeta = 1, 2\&, °F)$ using $CHFHWA$ operator that is given below:

$$Ʋ_{ɩ} = CHFHWA_{ƛ}(Ʋ_{ɩ\zeta}, Ʋ_{ɩ\zeta}, ., ., Ʋ_{ɩ°F}) = \underset{\zeta=1}{\overset{°F}{\oplus_H}} \left(\underline{ɷ}_\zeta\left(Ʋ_{ɩ\zeta}\right)\right)$$

$$= \bigcup_{\tilde{\chi}_{\overline{Ʊ}_{ɩ1}}, \tilde{Z}_{\overline{Ʊ}_{ɩ1}} \epsilon Ʋ_{ɩ1}, \dots, \tilde{\chi}_{\overline{Ʊ}_{ɩ°F}}, \tilde{Z}_{\overline{Ʊ}_{ɩ°F}} \epsilon Ʋ_{ɩ°F}} \left\{ \begin{array}{l} \left(\dfrac{\prod_{\zeta=1}^{°F}\left(1 + (ƛ - 1)\tilde{\chi}_{\overline{Ʊ}_\zeta}\right)^{\underline{ɷ}\zeta} - \prod_{\zeta=1}^{°F}\left(1 - \tilde{\chi}_{\overline{Ʊ}_\zeta}\right)^{\underline{ɷ}\zeta}}{\prod_{\zeta=1}^{°F}\left(1 + (ƛ - 1)\tilde{\chi}_{\overline{Ʊ}_\zeta}\right)^{\underline{ɷ}\zeta} + (ƛ - 1)\prod_{\zeta=1}^{°F}\left(1 - \tilde{\chi}_{\overline{Ʊ}_\zeta}\right)^{\underline{ɷ}\zeta}} \right) + \\ i\left(\dfrac{\prod_{\zeta=1}^{°F}\left(1 + (ƛ - 1)\tilde{Z}_{\overline{Ʊ}_\zeta}\right)^{\underline{ɷ}\zeta} - \prod_{\zeta=1}^{°F}\left(1 - \tilde{Z}_{\overline{Ʊ}_\zeta}\right)^{\underline{ɷ}\zeta}}{\prod_{\zeta=1}^{°F}\left(1 + (ƛ - 1)\tilde{Z}_{\overline{Ʊ}_\zeta}\right)^{\underline{ɷ}\zeta} + (ƛ - 1)\prod_{\zeta=1}^{°F}\left(1 - \tilde{Z}_{\overline{Ʊ}_\zeta}\right)^{\underline{ɷ}\zeta}} \right) \end{array} \right\}, (6)$$

On the other hand, the $CHFHWG$ operator:

$$\mathrm{Ʊ}_{\acute{\iota}} = CHFHWG_{\lambda}(\mathrm{Ʊ}_{\acute{\iota}\zeta}, \mathrm{Ʊ}_{\acute{\iota}\zeta}, .,.,., \mathrm{Ʊ}_{\acute{\iota}°F}) = \underset{\zeta=1}{\overset{°F}{\otimes_H}} \left(\mathrm{Ʊ}_{\acute{\iota}\zeta}\right)^{\underline{\omega}\zeta}$$

$$= \bigcup_{\tilde{\chi}_{\overline{\mathrm{U}}_{\acute{\iota}1}}, \tilde{Z}_{\overline{\mathrm{U}}_{\acute{\iota}1}} \epsilon \mathrm{Ʊ}_{\acute{\iota}1}, \dots, \tilde{\chi}_{\overline{\mathrm{U}}_{\acute{\iota}°F}}, \tilde{Z}_{\overline{\mathrm{U}}_{\acute{\iota}°F}} \epsilon \mathrm{Ʊ}_{\acute{\iota}°F}} \left\{ \begin{array}{l} \left(\dfrac{\lambda \prod_{\zeta=1}^{°F} \left(\tilde{\chi}_{\overline{\mathrm{U}}_{\zeta}}\right)^{\underline{\omega}\zeta}}{\prod_{\zeta=1}^{°F} \left(1 + (\lambda - 1)\left(1 - \tilde{\chi}_{\overline{\mathrm{U}}_{\zeta}}\right)\right)^{\underline{\omega}\zeta} + (\lambda - 1) \prod_{\zeta=1}^{°F} \left(\tilde{\chi}_{\overline{\mathrm{U}}_{\zeta}}\right)^{\underline{\omega}\zeta}} \right) + \\ i \left(\dfrac{\lambda \prod_{\zeta=1}^{°F} \left(\tilde{Z}_{\overline{\mathrm{U}}_{\zeta}}\right)^{\underline{\omega}\zeta}}{\prod_{\zeta=1}^{°F} \left(1 + (\lambda - 1)\left(1 - \tilde{Z}_{\overline{\mathrm{U}}_{\zeta}}\right)\right)^{\underline{\omega}\zeta} + (\lambda - 1) \prod_{\zeta=1}^{°F} \left(\tilde{Z}_{\overline{\mathrm{U}}_{\zeta}}\right)^{\underline{\omega}\zeta}} \right) \end{array} \right\}$$

iv. Find the score values for each $\mathrm{Ʊ}_{\acute{\iota}}(\acute{\iota} = 1, 2, \dots, \mathrm{ɰ})$.
v. Rank them using step IV.
vi. End.

4.2. Numerical Problem

A university offering a fully funded scholarship to a student, but the applicants are four, i.e. $\mathrm{ꝛ}_1, \mathrm{ꝛ}_2, \mathrm{ꝛ}_3, \mathrm{ꝛ}_4$ and four criteria are suggested for them that are U_1: academic performance, U_2: research contribution, U_3: extracurricular activities, U_4: communicational skill. After the evaluation, we get the best attribute or the best student who met the criteria. The weight vector is given as $\underline{\omega} = (0.4, 0.2, 0.3, 0.1)^{\mathcal{T}}$. Following this, we use the initial strategy to choose the best student.

i. Throughout the assessment, it is crucial that the university's board collaborates to choose their own preference for the alternatives $\mathrm{ꝛ}_{\acute{\iota}}$ satisfy U_{ζ}. Then the decision matrix $\bar{\mathcal{H}} = \left(\mathrm{Ʊ}_{\acute{\iota}\zeta}\right)_{\mathrm{ɰ}\times °F}$ Shown in Table.1.
ii. Suppose $\lambda = 0.3$, and integrate all the desired outcomes $\overline{\mathrm{U}}_{\acute{\iota}\zeta}(\acute{\iota} = 1, 2, \&, \mathrm{ɰ}; \zeta = 1, 2\&, °F)$ in row $\acute{\iota}$ of the decision matrix $\bar{\mathcal{H}}$, according to Eq.(6), into optimal values $\mathrm{Ʊ}_{\acute{\iota}}$ of alternatives $\mathrm{ꝛ}_{\acute{\iota}}(\acute{\iota} = 1, 2,3,4)$.
iii. Compute the score values for each $\mathrm{Ʊ}$.

$$\mathfrak{s}(\mathrm{Ʊ}_1)_{0.3} = 0.5224, \mathfrak{s}(\mathrm{Ʊ}_2)_{0.3} = 0.4486, \mathfrak{s}(\mathrm{Ʊ}_3)_{0.3} = 0.5280, \mathfrak{s}(\mathrm{Ʊ}_4)_{0.3} = 0.5680.$$

iv. By the result of step iii, we have $\mathfrak{s}(\mathrm{Ʊ}_4)_{0.3} \geq \mathfrak{s}(\mathrm{Ʊ}_3)_{0.3} \geq \mathfrak{s}(\mathrm{Ʊ}_1)_{0.3} \geq \mathfrak{s}(\mathrm{Ʊ}_2)_{0.3}$. Therefore, $\mathrm{U}_4 > \mathrm{U}_3 > \mathrm{U}_1 > \mathrm{U}_2$, indicating that the best alternative is U_4 . Furthermore, let $\lambda = 0.3, 0.6,3,6, 10$ which represent different preferences of decision makers regarding decision-making data. The relative score values and order of the alternatives are presented in Table 2. To highlight the result of various attitudinal character parameter λ on the AOs, we employ various values of $\lambda \in (0, 10]$, as interpreted by the experts. Changing trends of global preference score functions. $\mathfrak{s}(\mathrm{Ʊ}_{\acute{\iota}})(\acute{\iota} = 1, 2,3,4)$ In $CHFHWA$ with respect to λ illustrated Fig.1 we can see clearly from Figure1 that the score outcomes generated through the $CHFHWA$ operator has inversely proportional to the parameter λ. Choose the values of the parameter λ according to their preferences. As illustrated in Fig.1, for the $CHFHWA$ operator, the last ranking of the alternatives is unaffected for $\lambda \epsilon (0, 10]$.

Table 1

A CHFDM

	$\mathfrak{r}_1$	$\mathfrak{r}_2$	$\mathfrak{r}_3$	$\mathfrak{r}_4$
υ_1	$\begin{Bmatrix} 0.2 + i0.3, \\ 0.5 + i0.7 \end{Bmatrix}$	$\begin{Bmatrix} 0.4 + i0.2, 0.3 + i0.4, \\ 0.6 + i0.9 \end{Bmatrix}$	$\begin{Bmatrix} 0.1 + i0.6, \\ 0.8 + i0.5 \end{Bmatrix}$	$\begin{Bmatrix} 0.7 + i0.1, \\ 0.9 + i0.8 \end{Bmatrix}$
υ_2	$\begin{Bmatrix} 0.4 + i0.2, \\ 0.1 + i0.3 \end{Bmatrix}$	$\begin{Bmatrix} 0.7 + i0.1, \\ 0.3 + i0.4 \end{Bmatrix}$	$\begin{Bmatrix} 0.5 + i0.6, \\ 0.2 + i0.7 \end{Bmatrix}$	$\begin{Bmatrix} 0.6 + i0.9, \\ 0.3 + i0.5 \end{Bmatrix}$
υ_3	$\begin{Bmatrix} 0.3 + i0.5, \\ 0.9 + i0.6 \end{Bmatrix}$	$\begin{Bmatrix} 0.8 + i0.3, \\ 0.2 + i0.4 \end{Bmatrix}$	$\begin{Bmatrix} 0.4 + i0.2, \\ 0.1 + i0.1 \end{Bmatrix}$	$\begin{Bmatrix} 0.5 + i0.7, \\ 0.6 + i0.8 \end{Bmatrix}$
υ_4	$\begin{Bmatrix} 0.4 + i0.2, \\ 0.9 + i0.5 \end{Bmatrix}$	$\begin{Bmatrix} 0.5 + i0.3, \\ 0.3 + i0.7 \end{Bmatrix}$	$\begin{Bmatrix} 0.8 + i0.1, \\ 0.2 + i0.9 \end{Bmatrix}$	$\begin{Bmatrix} 0.1 + i0.6, 0.2 + i0.8, \\ 0.6 + i0.4 \end{Bmatrix}$

Table 2

Score values and ranking of the alternatives by using the CHFHWA operator.

	$CHFHWA_{0.3}$	$CHFHWA_{0.6}$	$CHFHWA_3$	$CHFHWA_6$	$CHFHWA_{10}$
υ_1	0.5567	0.5406	0.5022	0.4884	0.4828
υ_2	0.4486	0.4353	0.3998	0.3872	0.3800
υ_3	0.5280	0.5132	0.4734	0.4589	0.4505
υ_4	0.5929	0.5771	0.5387	0.5264	0.5197
Ranking	$\upsilon_4 > \upsilon_1 > \upsilon_3 > \upsilon_2$	$\upsilon_4 > \upsilon_1 > \upsilon_3 > \upsilon_2$	$\upsilon_4 > \upsilon_1 > \upsilon_3 > \upsilon_2$	$\upsilon_4 > \upsilon_2 > \upsilon_3 > \upsilon_2$	$\upsilon_4 > \upsilon_1 > \upsilon_3 > \upsilon_2$

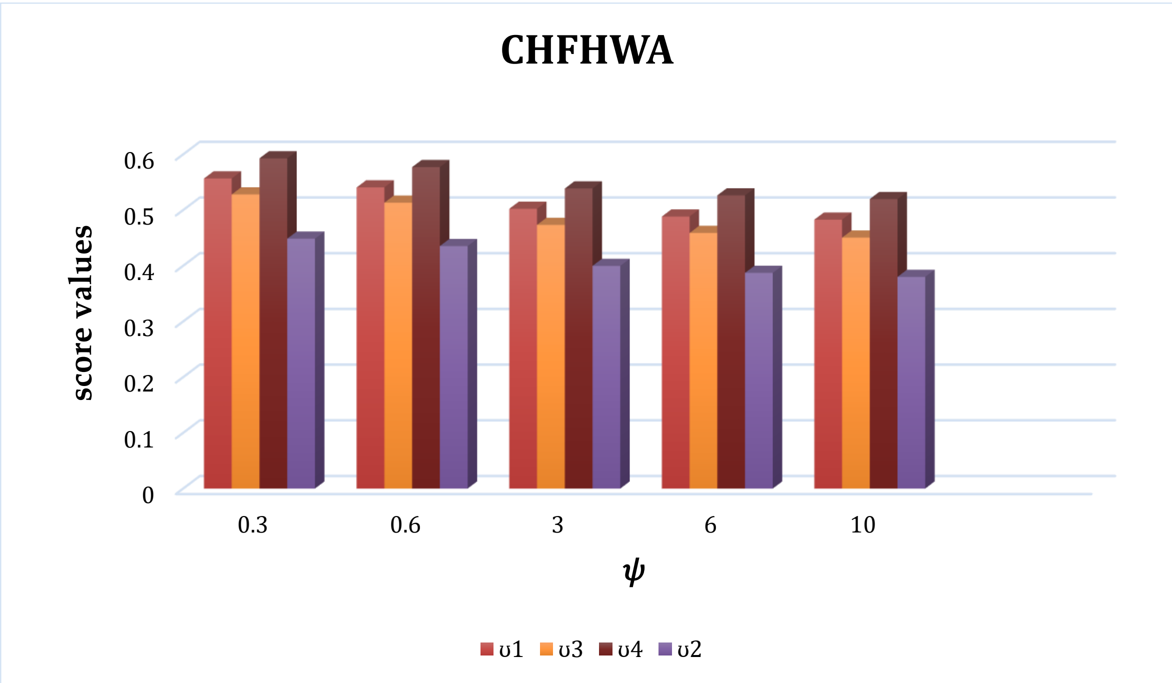

Fig. 1. The graphical interpretation of score values by using the CHFHWA operator.

Table 3

Score values and ranking of the alternatives by using the CHFHWG operator.

	$CHFHWG_{0.3}$	$CHFHWG_{0.6}$	$CHFHWG_{3}$	$CHFHWG_{6}$	$CHFHWG_{10}$
υ_1	0.3928	0.4063	0.4398	0.4503	0.4560
υ_2	0.3388	0.3219	0.3462	0.3533	0.3570
υ_3	0.3399	0.3564	0.3960	0.4090	0.4161
υ_4	0.4273	0.4358	0.4860	0.5046	0.5155
Ranking	$\upsilon_4 > \upsilon_1 > \upsilon_3 > \upsilon_2$	$\upsilon_4 > \upsilon_1 > \upsilon_3 > \upsilon_2$	$\upsilon_4 > \upsilon_1 > \upsilon_3 > \upsilon_2$	$\upsilon_4 > \upsilon_1 > \upsilon_3 > \upsilon_2$	$\upsilon_4 > \upsilon_1 > \upsilon_3 > \upsilon_2$

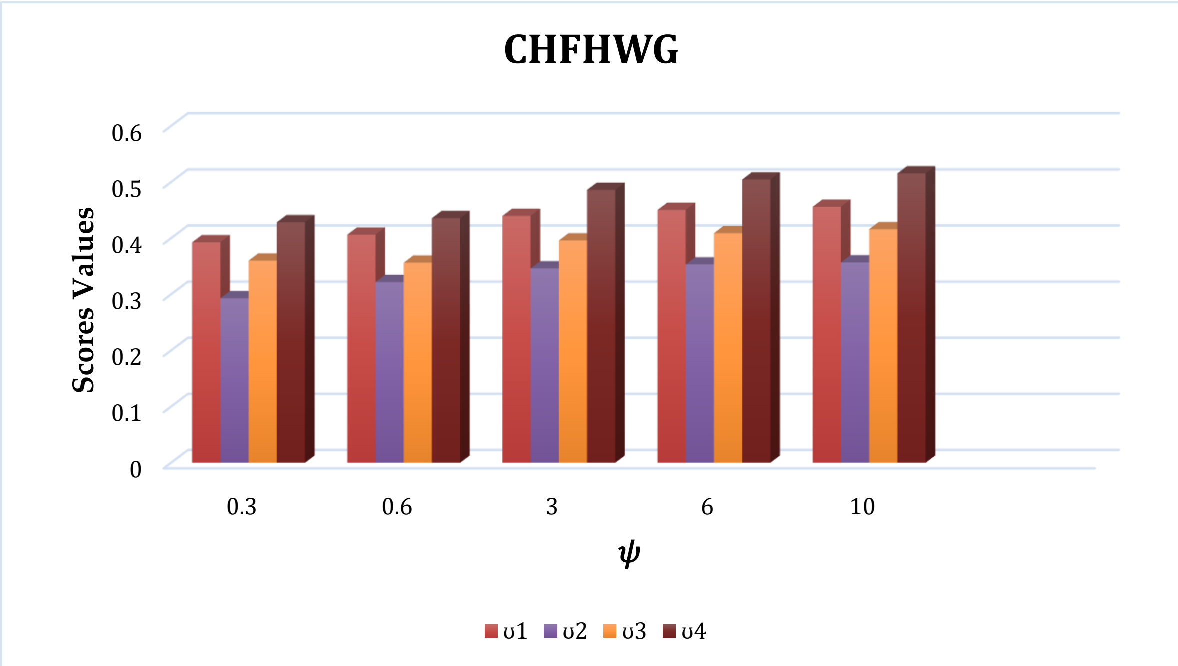

Fig. 2. The graphical interpretation of score values by using the CHFHWG operator.

When we replace $CHFHWA$ to $CHFHWG$ operator in step 2, and also comparing Table 2 with Table.3, then we found that, except for the case having the identical configuration parameter value ⅄ and aggregation argument, all the evaluation score values of $CHFHWG$ operator are less than the score values of $CHFHWA$ operator. In Fig.2, the global score functions $\mathfrak{s}(\mho_i)(i = 1,2,3,4)$ are shown to change with ⅄ in $CHFHWG$ operators. It is clear from Fig.2, the score values of $CHFHWG$ operator rise as the parameter ⅄ increase. The following are the rankings of alternatives in different parameter ⅄.

1. If ⅄$\epsilon(0, 2]$, then $\upsilon_4 > \upsilon_1 > \upsilon_3 > \upsilon_2$.
2. If ⅄$\epsilon(2,5]$, then $\upsilon_4 > \upsilon_1 > \upsilon_3 > \upsilon_2$.
3. If ⅄$\epsilon(5,8]$, then $\upsilon_4 > \upsilon_1 > \upsilon_3 > \upsilon_2$.
4. If ⅄$\epsilon(8, 10]$, then $\upsilon_4 > \upsilon_1 > \upsilon_3 > \upsilon_2$.

Hence, for the $CHFHWG$ operator, the parameter ⅄ value has an important effect on the score values and rank of alternatives. The analysis above indicates that while the $CHFHWA$ operator identifies the same top solution as the $CHFHWG$ operator.

5. Comparison Analysis

In this segment, we will compare the previous theories with our suggested work. It plays a key role in determining which theory is most advanced and best suited to the work discussed above. First, we will make a list of all previous theories and compare them with our proposed work.

1. Fuzzy Set Theory by Zadeh [4, 5].
2. Decision-making using HFS by Torra and Narukawa [8].
3. Decision-making using hesitant fuzzy information aggregation by Xia *et al.* [12].
4. Decision-making and fuzzy optimization using generalized OWA aggregation operators by Yager [14].
5. MADM using Dombi and Archimedean weighted operators for HFSs by Liu *et al.* [36].
6. MCDM using Hamacher AOs for HFSs by Tan *et al.* [28]
7. A new interpretation of complex membership grade by Tamir *et al.* [32].
8. Decision making with different innovative distance measures by Garg *et al.* [34].
9. Decision-making using circular bipolar complex fuzzy AOs by Senapati *et al.* [31].

Some of the above-listed theories are applicable only to fuzzy sets; others focus only on HFSs; and a few focus on hesitant aggregation operators. DAWs AOs relate to our proposed work, but they cannot handle the flexibility of complex numbers with Hamacher AOs. Now we develop Table 4, in which we rank the score values of all the above Example 4.2 and take $\lambda = 0.3$.

Table 4

Comparison between the proposed work and previous theories.

Theories	$\mathfrak{s}(\upsilon_1)$	$\mathfrak{s}(\upsilon_1)$	$\mathfrak{s}(\upsilon_1)$	$\mathfrak{s}(\upsilon_1)$	Ranking
Zadeh [4, 5]	×××	×××	×××	×××	×××
Torra and Narukawa [8]	×××	×××	×××	×××	×××
Xia *et al.* [12]	×××	×××	×××	×××	×××
Yager [14]	×××	×××	×××	×××	×××
Liu *et al.* [36]	×××	×××	×××	×××	×××
Tan *et al.* [28]	×××	×××	×××	×××	×××
Tamir *et al.* [32]	×××	×××	×××	×××	×××
Garg *et al.* [34]	×××	×××	×××	×××	×××
Senapati *et al.* [31]	×××	×××	×××	×××	×××
Proposed Work (CHFHWA)	0.5667	0.4486	0.5280	0.5929	$\upsilon_4 > \upsilon_1 > \upsilon_3 > \upsilon_2$
Proposed Work (CHFHWG)	0.3929	0.3388	0.3399	0.4273	$\upsilon_4 > \upsilon_1 > \upsilon_3 > \upsilon_2$

In Table 4 above, it is clear that all the theories failed in the face of our proposed work. The theories proposed by Tan *et al.* [28] and Liu *et al.* [37] are suitable, but they do not give the concept of CFS. The theory proposed by Senapati *et al.* [31] was much better because it introduced the concept of complex fuzzy sets, but it does not define CHFS. It is clear that our proposed work is superior to the above, as it addresses the concept of CHFS using Hamacher aggregation-weighted operators.

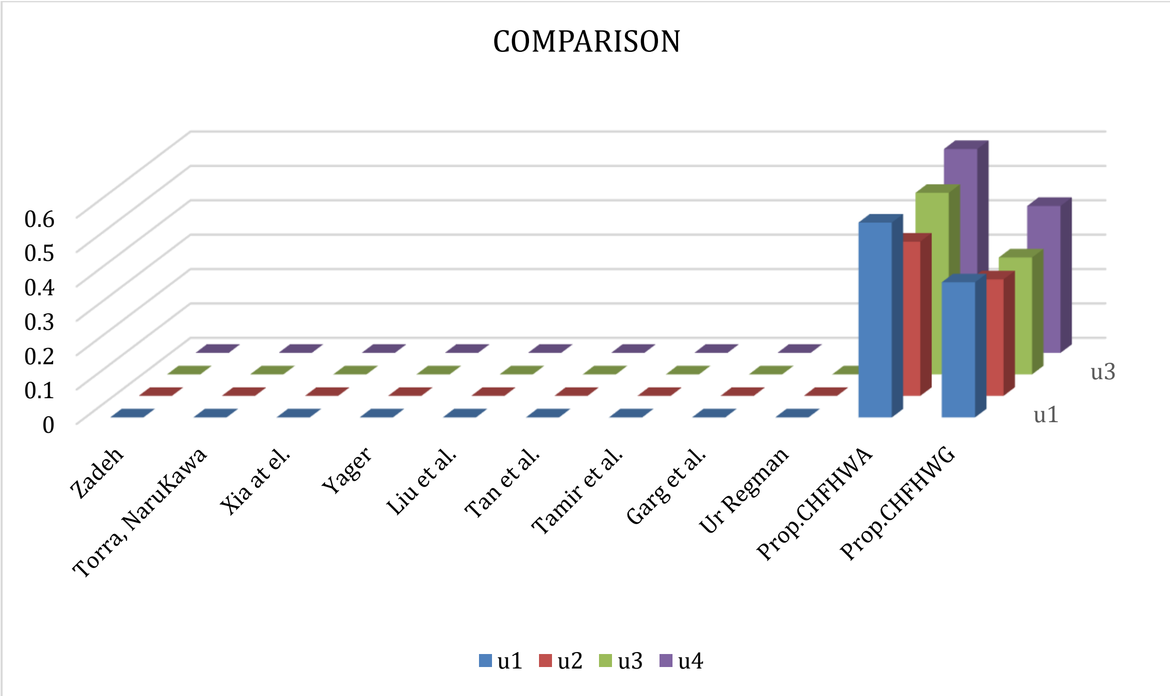

Fig. 3. Comparison chart

6. Conclusions

This article developed a new generalization of Hamacher AOs within the CHFS framework. The CHFHWA, CHFHWG, GCHFHWA, and GCHFHWG operators we have constructed provide a mathematically sound and flexible tool for addressing uncertainty and hesitation in MCDM. These operators combine the advantages of Hamacher t-norms with the expressiveness of CHFS, providing greater flexibility for integrating complex and inexact information. The practical value of the proposed operators was demonstrated by their application to a real-life decision-making problem. The scholarship selection case study demonstrated that the operators provide consistent and meaningful rankings of the alternatives, even when the attitudinal parameters are varied. Moreover, the comparative analysis revealed that previous models do not offer the same degree of flexibility and accuracy, thereby demonstrating the excellence of our method. This contribution not only fills a clear research gap in the literature but also provides a basis for future developments in CHFS-based decision-making. The suggested framework can be scaled to address large-scale group decision-making, uncertainty modeling in engineering and healthcare, and decision-making in new areas of data sciences and artificial intelligence. We aim to expand our study to various uncertain environments, such as probabilistic hesitant fuzzy sets [37], Pythagorean hesitant fuzzy sets [38], circular intuitionistic fuzzy sets [39] complex dual hesitant fuzzy sets [40], hesitant multi-fuzzy soft sets [41] and T-spherical fuzzy sets [42] and complex picture fuzzy sets [43].

Author Contributions

Conceptualization, M.H.F. and U.R.; methodology, M.H.F.; software, J.A.; validation, M.R.A., T.M., and U.R.; formal analysis, M.R.A.; investigation, J.A.; resources, J.A. and T.M.; data curation, J.A., and U.R.; writing—original draft preparation, M.H.F.; writing—review and editing, M.R.A.; visualization, M.H.F.; supervision, T.M.; project administration, T.M.; funding acquisition, U.R. All authors have read and agreed to the published version of the manuscript.

Conflicts of Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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