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Structural Properties of N-Fuzzy Graphs with Applications to Bridges, Cut Nodes, and Trees

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ABSTRACT

In this paper, we study several structural properties of N-fuzzy graphs. Fundamental concepts, including N-fuzzy sets, N-fuzzy relations, N-fuzzy bridges, N-fuzzy cut nodes, N-fuzzy cycles, and N-fuzzy trees, are reviewed and analyzed. The relationships among these concepts are investigated through theoretical results and illustrative examples. In particular, we examine conditions involving N-fuzzy bridges and N-fuzzy cut nodes and provide counterexamples showing that certain converse statements do not hold in general. Furthermore, properties of N-fuzzy trees are discussed and characterized using the strongest paths and the maximum spanning trees. The results obtained contribute to understanding the structural behavior of N-fuzzy graphs and provide a basis for future research in N-fuzzy network analysis and related applications

1. Introduction

Graph theory is a fundamental branch of discrete mathematics that provides powerful tools for modeling relationships among objects and entities. Owing to its extensive applications in communication networks, transportation systems, social sciences, computer science, and engineering, graph theory has become one of the most actively studied areas of mathematics. Comprehensive treatments of graph-theoretic concepts and applications can be found in the works of Harary [5], Bollobás [3], Balakrishnan [1], Harris [6], and Reinhard [9].

In many practical situations, the relationships among objects cannot be precisely described due to uncertainty, vagueness, and incomplete information. To address such issues, Zadeh [16] introduced the concept of fuzzy sets in 1965, which has since become a fundamental framework for uncertainty modeling. The integration of fuzzy set theory with graph theory led to the development of fuzzy graphs by Rosenfeld [10,11]. In a fuzzy graph, vertices and edges are associated with membership values, thereby providing a flexible representation of uncertain relationships.

Since the introduction of fuzzy graphs, numerous researchers have contributed to their theoretical development. Cerruti [4] investigated the relationship between graph theory and fuzzy graph structures, while Takeda [13] studied connectivity properties in fuzzy graphs. Yeh and Bang

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[14] discussed fuzzy relations and their applications in clustering analysis. Bhattacharya [2] established several important properties of fuzzy graphs and provided foundational results that stimulated further developments in fuzzy graph theory. Subsequently, Mordeson and Nair [7,8] made significant contributions through their studies of fuzzy cycles, cocycles, and fuzzy hypergraphs, establishing a systematic framework for the analysis of fuzzy graph structures.

The investigation of tree structures has also played a central role in fuzzy graph theory. Sunitha and Vijayakumar [12] introduced important characterizations of fuzzy trees and examined several structural properties associated with connectivity and spanning structures. Similarly, Yuan and Shaw [15] demonstrated the usefulness of fuzzy structures in decision-tree-based systems, further emphasizing the applicability of fuzzy graph concepts in uncertain environments. Motivated by the success of fuzzy graph theory, several generalizations and extensions have been proposed to model more complex forms of uncertainty. Among these extensions, N-fuzzy sets provide an alternative framework in which membership values are defined on the interval $[-1,0]$. Based on N-fuzzy sets and N-fuzzy relations, the notion of N-fuzzy graphs has been introduced as a generalization of classical fuzzy graphs. The resulting framework extends several graph-theoretic concepts and provides an alternative approach for analyzing structures characterized by negative-valued memberships. Although a considerable amount of research has been devoted to fuzzy graphs and their extensions, several structural properties of N-fuzzy graphs remain insufficiently explored. In particular, the relationships among N-fuzzy bridges, N-fuzzy cut nodes, strongest paths, and N-fuzzy trees require further investigation. Moreover, the validity of certain converse statements associated with these concepts has not been fully examined. Such issues are important for understanding the structural behavior of N-fuzzy graphs and for extending classical graph-theoretic results to N-fuzzy environments.

Recent developments in fuzzy mathematics have demonstrated the importance of generalized fuzzy structures in decision-making, optimization, intelligent systems, and uncertainty modeling [25]. Studies involving intuitionistic fuzzy sets, hesitant fuzzy models, and Pythagorean fuzzy frameworks have shown promising applications in medical diagnosis, supply chain optimization, assignment problems, and decision-support systems [22–24,19]. Furthermore, contemporary applications of intelligent systems and fuzzy methodologies have been reported in artificial intelligence, image analysis, deep learning, and multi-agent systems [17,20,21]. These developments indicate that advanced fuzzy graph structures may provide useful tools for modeling uncertainty in complex networked systems.

Motivated by these observations, the present paper investigates several structural properties of N-fuzzy graphs with particular emphasis on N-fuzzy bridges, N-fuzzy cut nodes, and N-fuzzy trees. We establish characterizations based on the strongest paths and maximum spanning trees, analyze the relationships among important structural concepts, and provide counterexamples demonstrating that certain converse statements do not hold in general. The results obtained contribute to a better understanding of N-fuzzy graph theory and provide a foundation for future investigations involving uncertainty-aware network analysis.

1.1 Contributions of the Paper

The main contributions of this work are summarized as follows:

i. Fundamental concepts related to N-fuzzy graphs, including N-fuzzy bridges, N-fuzzy cut nodes, N-fuzzy cycles, and N-fuzzy trees, are reviewed and analyzed.
ii. Characterizations of N-fuzzy bridges and N-fuzzy cut nodes based on strongest paths and maximum spanning trees are discussed.

iii. The relationship between N-fuzzy bridges and N-fuzzy cut nodes is investigated through several illustrative examples.
iv. Counterexamples are presented to demonstrate that certain converse statements concerning N-fuzzy cut nodes and N-fuzzy trees do not hold in general.
v. The structural behavior of N-fuzzy trees is studied, and several related properties and characterizations are established.
vi. Theoretical results concerning the strongest paths, N-fuzzy bridges, cut nodes, and maximum spanning trees are examined to provide a better understanding of the structural properties of N-fuzzy graphs.

1.2 Organization of the Paper

The remainder of this paper is organized as follows. Section 2 presents the basic definitions and preliminary concepts related to N-fuzzy graphs. The notions of N-fuzzy bridges and N-fuzzy cut nodes are investigated, and several characterizations based on strongest paths and maximum spanning trees are established. Illustrative examples and remarks are also provided to highlight important structural properties. Section 3 is devoted to the study of N-fuzzy trees. Various properties and characterizations of N-fuzzy trees are discussed, and their relationships with N-fuzzy bridges, cut nodes, strongest paths, and spanning trees are analyzed. Furthermore, examples and counterexamples are presented to clarify the validity and limitations of certain theoretical results. Finally, Section 4 concludes the paper by summarizing the main findings and outlining possible directions for future research in N-fuzzy graph theory and related areas.

2. Preliminaries

Definition 2.1
An N-fuzzy set f (N-fuzzy subset of non-empty set S) is defined as mapping.
$f\colon \mathrm{S} \to [-1,0]$
Where f(x) is the negative membership degree of x to the N-fuzzy set S. We denote by NF (S) the collection of all N-fuzzy subsets of S.
The negative-valued memberships in an N-fuzzy graph may be interpreted as representing adverse, undesirable, or negatively weighted relationships among vertices and edges. Depending on the application, these values can model opposition, vulnerability, communication loss, or other forms of negative influence within a network.
Definition 2.2 Let S and T be two non-empty sets, and let f and g be N-fuzzy subsets of S and T, respectively. Then an N-fuzzy relation α of S×T is a relation such that α (x, y) $\geq f(\mathrm{x}) \vee g(\mathrm{y})$, ∀ x ∈ S and y ∈ T.
Definition 2.3 Let S be a non-empty set. An N-fuzzy graph is a pair of functions. G: $(f,\ \alpha)$, where f is an N-fuzzy subset of a non-empty set S and α is a symmetric N-fuzzy relation on f. *i.e.*
$f\colon \mathrm{S} \to\ [-1, 0]$ and $\alpha\colon \mathrm{S} \times \mathrm{S}\ \to\ [-1, 0]$ such that $\alpha(\mu,\ \nu) \geq f(\mu) \vee\ f(\nu)\ \ \forall\ \ \mu\ ,\ \nu \in \mathrm{S}$.
Definition 2.4 An arc (μ, ν) is an N-fuzzy bridge of G: $(f,\ \alpha)$ if the deletion of (, ν) reduces the strength of connectedness between the same pair of nodes. Equivalently, (μ, ν) is an N-fuzzy bridge if and only if there are nodes x, y such that (μ, ν) is an arc of every strongest x-y path.
Definition 2.5 A node is an N-fuzzy cut node of G: $(f,\ \alpha)$ if removal of it reduces the strength of connectedness between same other pair of nodes. Equivalently, w is an N-fuzzy cut node if and only if there are u and v distinct from w such that w is on every strongest u-v path.

Example 2.1 In Figure 2.1 (u_1, u_3), (u_3, u_2), (u_2, u_4), (u_4, u_5) are the N-fuzzy bridges and u_2, u_3, u_4 are the N-fuzzy cut nodes of an N-fuzzy graph G: (f, α).

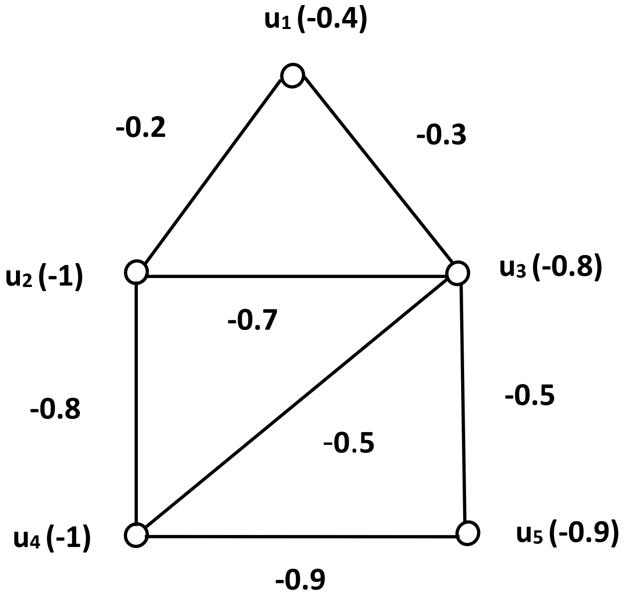

Fig. 1 An N-fuzzy graph illustrating an N-fuzzy bridge and an N-fuzzy cut node.

Definition 2.6 A connected N-fuzzy graph G: (f, α) is an **N-fuzzy tre**e if it has an N-fuzzy spanning sub graph F: (f, β), which is a tree, where for all arcs (u, v) not in F $\alpha(u, v) > \alpha^{\infty}(u, v)$. Equivalently, there is a path in F between u and v whose strength exceeds $\alpha(\mu, \nu)$ for all (μ, ν) not in F. If G is such that G^* is a tree, then F is G itself.

Example 2.2

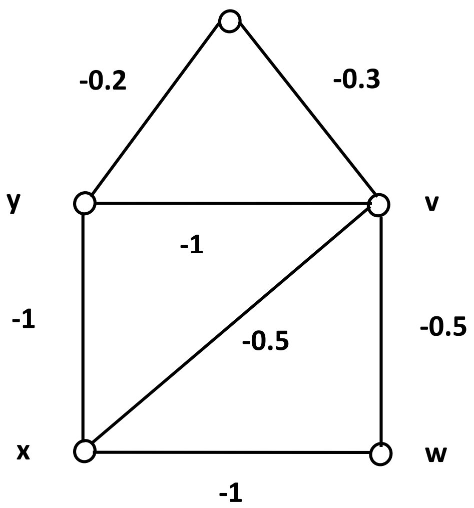

Fig.2. An example of an N-fuzzy graph containing an N-fuzzy bridge whose removal alters the connectivity structure of the graph.

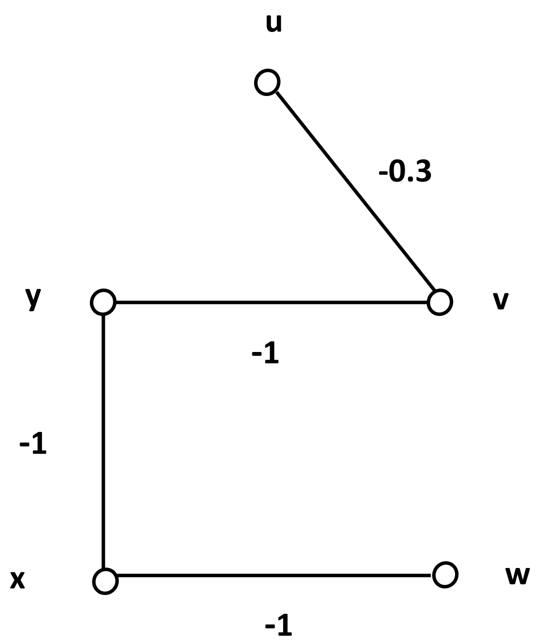

Fig.3. Spanning subgraph F: (f, β)

Here α (u, y) = -0.2 > -0.3 = $\beta^{\propto}\ (u, y)$, $\alpha\ (v, w)\ =\ -0.5\ >\ -1\ =\ \beta^{\propto}\ (v, w)$ and $\alpha\ (v, x)\ =\ -0.5\ >\ -1\ =\ \beta^{\propto}(v, x)$.

Definition 2.7 Let G: (f, α) be an N-fuzzy graph such that G^* is cycle. Then, G is called an N-fuzzy cycle if it has more than one weakest arc.

Definition 2.8 A connected N-fuzzy graph G: (f, α) with no N-fuzzy cut nodes is called a block.

In [3, 4] it was noted that blocks in N-fuzzy graphs may have N-fuzzy bridges. In the next example, G1 is a block without an N-fuzzy bridge, and G_2 is a block with N-fuzzy bridges (u_1, u_2) and (u_3, u_4).

Example 2.3

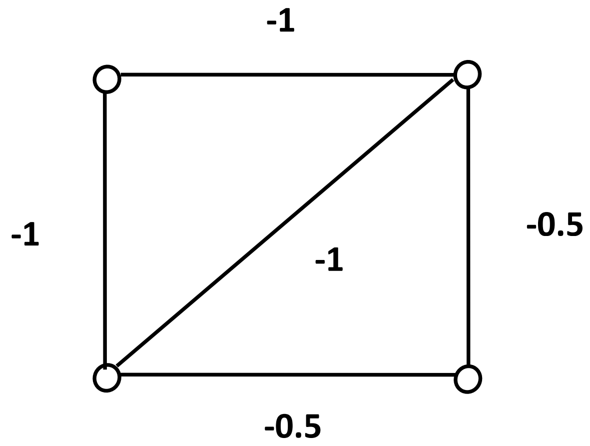

Fig.4. An N-fuzzy graph showing an N-fuzzy cut node whose removal affects the connectedness of the graph.

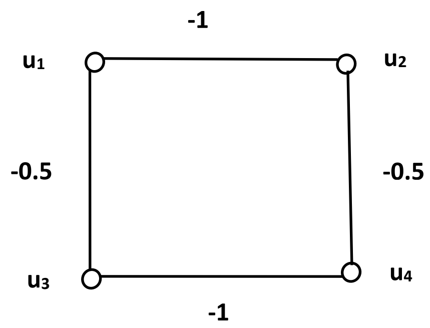

Fig.5. An N-fuzzy graph showing an N-fuzzy cut node whose removal affects the connectedness of the graph.

Theorem 2.1 An arc (a node) is an N-fuzzy bridge (N-fuzzy cut node), if and only if there exists a partition V of nodes into subsets U, W, such that all nodes u ∈ U and w ∈ W, the arc (the node) is on every strongest u-v path.
Theorem 2.2 Let G: (f, α) be an N-fuzzy graph with $U = u_1, u_2, \dots, u_n$. And let C be the cycle $u_1, u_2, \dots \dots u_n \ , u_1$. If $C \subseteq \alpha^*$ and for every arc $(u_j, u_k) \in \alpha^*$- C, $\alpha\ (u_j, u_k) < Max\{\alpha\ (u_i, u_{i+1}): i = 1,2,3,4, \dots, n\ \}$ where, $u_{n+1} = u_1$, then either α is a constant on C or G has an *N*-fuzzy bridge.
Theorem 2.3 Let G be a connected N-fuzzy graph. If there is one strongest path between any two nodes of G, then G is called an N-fuzzy tree.
Theorem 2.4 If G is an N-fuzzy tree, then arcs of α are the N-fuzzy bridges of G.
Theorem 2.5 The following statements are equivalent for an arc (u, v) of an N-fuzzy graph G: (f, α).
(1) (u, v) is an N-fuzzy bridge.
(2) (u, v) is not the weakest arc of any cycle in G.
Theorem 2.6 An arc (a node) is an N-fuzzy bridge (N-fuzzy cut node), if and only if there exists a partition V of nodes into subsets U, W, such that all nodes u ∈ U and w ∈ W, the arc (the node) is on every strongest u-v path.
Theorem 2.7 Let G: (f, α) be an N-fuzzy graph with $U = u_1, u_2, \dots \dots, u_n$. And let C be the cycle $u_1, u_2, \dots \dots u_n \ , u_1$. If $C \subseteq \alpha^*$ and for every arc $(u_j,\ u_k) \in \alpha^*$- C, $\alpha\ (u_j, u_k) < \mathrm{Max}\{\alpha\ (u_i, u_{i+1}): i = 1,2,3,4, \dots \dots, n\ \}$ where, $u_{n+1} = u_1$, then either α is a constant on C or G has an *N*-fuzzy bridge.
Theorem 2.8 Let G be a connected N-fuzzy graph. If there is one strongest path between any two nodes of G, then G is called an N-fuzzy tree.
Theorem 2.9 If G is an N-fuzzy tree, then arcs of α are the N-fuzzy bridges of G.
Theorem 2.10 Let $G: (f,\ \alpha)$ be an N- fuzzy graph and (u, v) be an N-fuzzy bridge of G. Then, $\alpha^{\propto}$(u, v) = α (u, v).
Proof: Suppose that (u, v) is an N-fuzzy bridge and that $\alpha^{\propto}$(u, v) exceeds α (u, v). Then there exists a strongest u-v Path with strength greater than α (u, v), and all arcs of this strongest path have strength greater than α(u, v). Now, this path to gather with the arc (u,
Remark 2.1 It follows that an arc (u, v) is an N-fuzzy bridge if and only if it is the unique strongest u-v path. However, the converse of this theorem is not true.
In the following N-fuzzy graph (Fig. 2 .4), (u_2, u_4) and (u_3, u_4) are the only N-fuzzy bridges and $\alpha^{\propto}(u_1, u_2) = \alpha\ (u_1, u_2) = 0.4 = \alpha^{\propto}(u_1, u_3) = \alpha\ (u_1, u_3)$ but (u_1, u_2) and (u_1, u_3) are not N-fuzzy bridges. The condition for the converse to be true is discussed next.

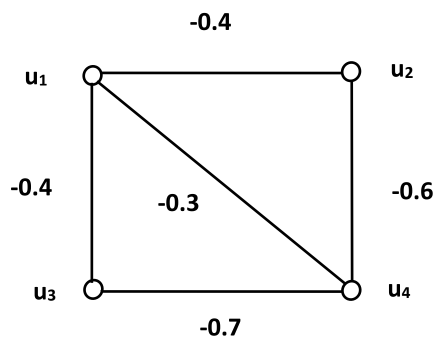

Fig.6. Illustration of the strongest paths between vertices in an N-fuzzy graph.

We first observe that identifying N-fuzzy cut nodes is not easy. In the next Theorem, we characterize N-fuzzy cut nodes in G such that G* is a cycle and present a sufficient condition for a node to be a fuzzy cut node in the general case.

Theorem 2.11 Let $\mathrm{G}\colon (f,\ \alpha)$ be an N-fuzzy graph such that G* is a cycle. Then, a node is an N-fuzzy cut node of G if and only if it is a common node of two N-fuzzy bridges.

Proof: Let w be an N-fuzzy cut node of G. Then, there exist u and v, distinct from w, such that w is on every strongest u-v path, which is unique since G* is a cycle, and it follows that all its arcs are N-fuzzy bridges. Thus, we are a common node of two N-fuzzy bridges.

Conversely, let w be a common node of two N-fuzzy bridges (u, w) and (w, v). Then both (u, w) and (u, v) are not the weakest arcs of G. Also, the path from u to v not containing the arcs (u, w) and (w, v) has strength less than α (u, w) $\vee$ α (w, v). Thus, the strongest u-v path is the path u, w, v, and $\alpha^{\propto}$(u, v) = α (u, w) $\vee$ α (w, v). Thus, w is an N-fuzzy cut node.

In general, we have,

Theorem 2.12 Let $\mathrm{G}\colon (f,\ \alpha)$ be an N-fuzzy graph and let w be a common node of two N-fuzzy bridges. Then w is an N-fuzzy cut node.

Proof: Let (u_1, w) and (w, u_2) be two N-fuzzy bridges. Then there exists some u, v such that (u_1, w) is on every strongest u-v path. If w is distinct from u and v, it follows that w is an N-fuzzy cut node. Next, suppose one of v, u is w so that (u_1, w) is on every strongest u-w path or (w, u_2) is on every strongest w-v path. If possible, let w not be an N-fuzzy cut node. Then, between every two nodes, distinct from w, there exists at least one strong path not containing w. There exists a strongest path P, joining u_1 and u_2, that does not contain w. This path, together with (u_1, w) and (w, u_2) forms a cycle. There are two cases.

Case: 1 Let u_1, w, and u_2 be not the strongest path. Then, either (u_1, w) or (w, u_2) or both become the weakest arcs of the cycle, which contradicts the fact that (u_1, w) and (w, u_2) are N-fuzzy bridges.

Case: 2 Let u_1, w, and u_2 be a strongest path joining u_1 to u_2. Then, $\alpha^{\propto}$ (u_1, u_2) = α (u_1, w) $\vee$ α (w, u_2) the strength of P. Thus, arcs of P are as strong as α (u_1, w) and α (w, u_2) which implies that (u_1, w), (w, u_2), or both are the weakest arcs of the cycle, contradicting the assumption.

Remark 2.2 The condition in the above theorem is not necessary. In Fig. 2.5, w is the N-fuzzy cut node; (u, w) and (v, x) are the only N-fuzzy bridges, in Fig. 2.6, w is the fuzzy cut node (u, w) is the only N-fuzzy bridge and (u, w) is the only N-fuzzy bridge. In Fig. 2 .7, w is the N-fuzzy cut node, and no arc is a fuzzy bridge. But the converse of Theorem 2.12 holds in N-fuzzy trees.

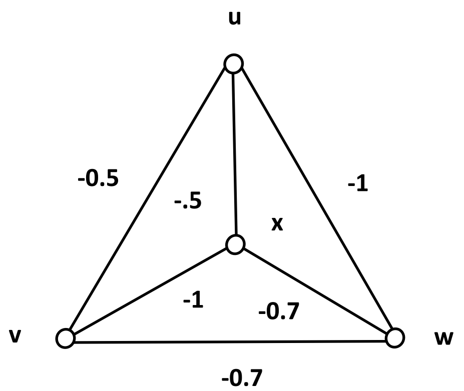

Fig.7. Illustration of N-fuzzy bridges

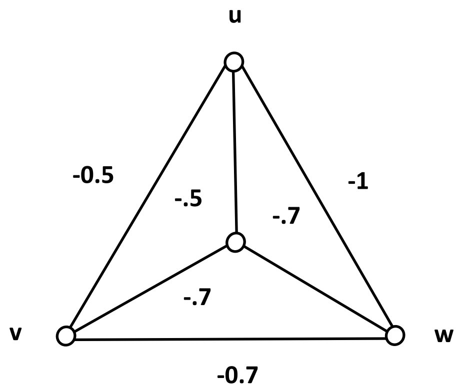

Fig.8. Illustration of N-fuzzy cut nodes

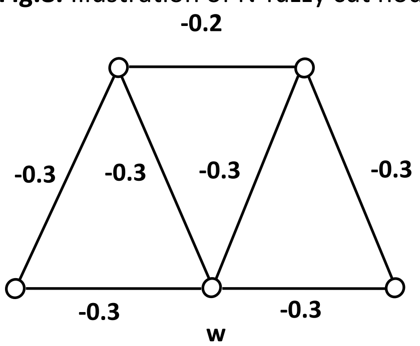

Fig.9. Illustration of no arc in N-fuzzy bridges

The counterexamples presented above indicate that the relationship between N-fuzzy bridges and N-fuzzy cut nodes is more intricate than its counterpart in classical graph theory. In particular, the existence of an N-fuzzy cut node does not necessarily imply the existence of two incident N-fuzzy bridges. These examples highlight the importance of additional structural conditions when extending classical graph-theoretic concepts to the N-fuzzy environment.

Remark 2.3 As distinct from crisp graph theory, there are N-fuzzy graphs with an N-fuzzy bridge and having no N-fuzzy cut node. Thus (u, w) and (v, x) are the N-fuzzy bridges, and no node is an N-fuzzy cut node, where $0 > a > b \geq -1$.

Lemma 2.1 If G: $(f,\ \alpha)$ is a complete N-fuzzy graph, then $\alpha^{\propto}$ (u, v) = α(u, v).

Lemma 2.2 A complete N-fuzzy graph has no N-fuzzy cut nodes [6].

Remark 2.4 From Lemma 2.1, we have in a complete N-fuzzy graph that each (u, v) is the strongest u-v path. But the converse does not hold, as we see in Figure 2.8 Graph G. Also, it follows from Lemma 3.2 that if in G: $(f,\ \alpha)$, $\alpha^{\propto}$ (u, v) = α (u, v) for all u, v, then G has no N-fuzzy cut nodes.

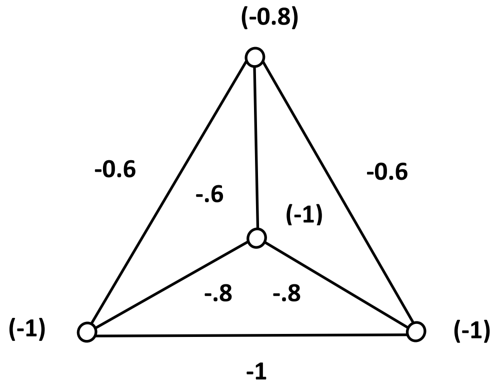

Fig.10. No N-fuzzy cut node in Graph G

Note that an N-fuzzy graph with an N-fuzzy bridge need not have N-fuzzy cut nodes, and a complete N-fuzzy graph has no N-fuzzy cut nodes [Lemma 2.2]. But we have,

Theorem 2.13 A complete N-fuzzy graph has at most one N-fuzzy bridge.

Proof: Let $G: (f,\ \alpha)$ be a complete N-fuzzy graph with |U| = 3. Then G can have at most one N-fuzzy bridge by Theorem 3.3 and Lemma 3.2. Now, let $|U| \leq 4$ and let u_1, u_2, u_3, and u_4 be any four nodes of G. Without loss of generality, let u_1 be such that $f(u_1)$ is the least among $f(u_1)$ `s, i = 1,2,3,4. Then (u_1, u_2), (u_1, u_3), and (u1, u4) are not N-fuzzy bridges; they are the weakest arcs of some cycle in the N-fuzzy subgraph induced by. u_1, u_2, u_3, and u_4. Now the arcs (u_2, u_3), (u_2, u_4), and (u_3, u_4) are adjacent to each other, and it follows that at most one of them can be an N-fuzzy bridge.

Theorem 2.14 Let $G: (f,\ \alpha)$ be a complete N-fuzzy graph with |U| = n. Then G has an N-fuzzy bridge if and only if there exists a decreasing sequence $\{t_1, t_2, t_3, \ldots t_{n-1}, t_n\}$ such that $t_n \leq t_{n-1} \leq t_{n-2}$, where $t_i = f(u_i)$. Also, the arc (u_{n-1}, u_n) is the N-fuzzy bridge of G.

Proof: Suppose that $G: (f, \alpha)$ is a complete N-fuzzy graph, and that G has an N-fuzzy bridge (u, v). Now α (u, v) = f (u) ∨ f (v) without loss of generality, let f (u) ≥ f (v) so that α (u, v) = f (u). Also note that (u, v) is not the weakest arc of any cycle in G. Now required to prove that f (u) < f f (w) ∀ w≠ v. In the country, assume that there is at least one node w≠v such that f (u) ≥ f (w). Now consider the cycle C: u, v, w, u. Then α (u, v) = α (u, w) = f (u) and

$$\alpha\,(u, w) = \begin{cases} f\,(v), & \text{if } f\,(u) = f(v) \text{ or if } f\,(u) > f\,(v) \geq f\,(w) \\ f\,(w) & \text{if } f\,(u) > f\,(w) > f\,(v) \end{cases}$$

In either case, the arc (u, v) becomes the weakest arc of the cycle, which contradicts our assumption that (u, v) is an N-fuzzy bridge.

Conversely, let $t_1 \geq t_2 \geq t_3 \geq \ldots\ldots \geq t_{n-1} \geq t_n$ and $t_i = f(u_i)$ ∀ i

Claim: Arc α (u_{n-1}, u_n) is an N-fuzzy bridge of G.

Now α (u_{n-1}, u_n) = f (u_{n-1}) ∨ f (u_n) = f (u_{n-1}) and clearly by hypothesis, all other arcs of G will have strength strictly less than f (u_{n-1}). Thus, the arc (u_{n-1}, u_n) is not the weakest arc of any cycle in G and hence is the N-fuzzy bridge.

Example 2.4 G_1 and G_2 (Fig.2.9 a and b) are complete fuzzy graphs where G_1 has no N-fuzzy bridges. The increasing sequence $\{t_i\}$ in G_2 is -.2, - .5, - 1, -1} and (u_3, u_4) is the N-fuzzy bridge.

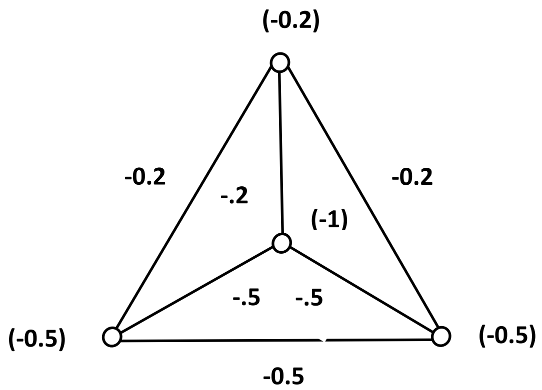

Fig.11. G_1 a complete N-fuzzy Graph

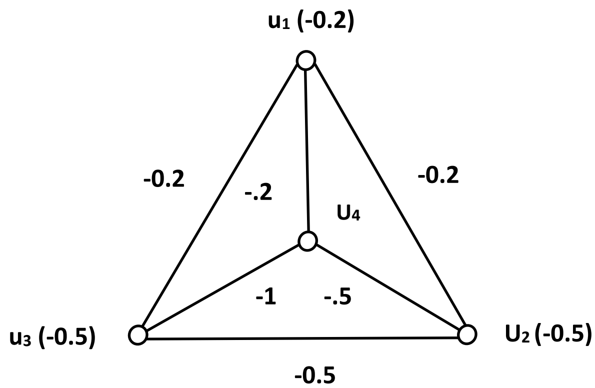

Fig.12. G_2 a complete N-fuzzy Graph

Now using the concept of maximum spanning tree of an N-fuzzy graph, we present a characterization of N-fuzzy bridge and N-fuzzy cut node. Also, in a (crisp) graph G*, note that each spanning tree is a maximum spanning tree. The following are characterizations of N-fuzzy bridge and N-fuzzy cut node, which are obvious in the crisp case.

Theorem 2.15 An arc (u, v) is an N-fuzzy bridge of $\mathrm{G}: (f,\ \alpha)$ if and only if (u, v) is in every maximum spanning tree of G.
Proof: Let (u, v) be an N- fuzzy bridge of G, then arc (u, v) is the unique strongest u- v path and hence is in every maximum spanning tree of G.
Conversely, let (u, v) be in every maximum spanning tree T of G and assume that (u, v) is not an N-fuzzy bridge. Then (u, v) is a weakest arc of some cycle in G and α^{∞} (u, v) $<$ α (u, v), which implies that (u, v) is in no maximum spanning tree of G.
Remark 2.5 From Theorem 2.15, it follows that arcs not in T are not N-fuzzy bridges of G and we have,
Corollary: If $\mathrm{G}: (f,\ \alpha)$ is a connected N-fuzzy graph with |U| = n then G has at most n-1 N-fuzzy bridge.
Theorem 2.16 A node w is an N-fuzzy cut node of $\mathrm{G}: (f,\ \alpha)$ if and only if w is an internal node of every maximum spanning tree of G.
Proof: Let w be an N-fuzzy cut node of G. Then there exist u, v distinct from w such that w is on every strongest u-v path. Now, each maximum spanning tree of G contains a unique strongest u-v path, and hence w is an internal node of each maximum spanning tree of G.

Conversely, let w be an internal node of every maximum spanning tree. Let T be a maximum spanning tree, and let (u, w) and (w, v) be arcs in T. Note that the path u, w, v is the strongest u-v path in T. If possible, assume that w is not an N-fuzzy cut node. Then, between every pair of nodes u

and v, there exists at least one strongest u-v path not containing w. Consider one such u-v path P, which clearly contains arcs not in T. Now, without loss of generality, let α^{∞} (u, v) = α (u, w) in T. Then arcs in P have strength $\leq \alpha$(u, w). Removal of (u, w) and adding P in T will result in another maximum spanning tree of G for which w is an end node, which contradicts our assumption.
Remark 2.6. It follows from Theorem 2.16 that the end nodes of the maximum spanning tree T of G are not N-fuzzy cut nodes of G. This results in the following corollary.

Corollary: [13] Every N-fuzzy graph has at least two nodes that are not N-fuzzy cut nodes of G. However, we see that there are N-fuzzy graphs with diametrical nodes, nodes which have maximum eccentricity, as N-fuzzy cut nodes, distinct from crisp graph theory.

2.1 Application Perspective of N-Fuzzy Graphs

N-fuzzy graphs can be employed to model networks in which negative-valued memberships represent adverse effects, risk levels, reliability degradation, or undesirable relationships. For example, in a communication network, vertices may represent communication devices, and edge memberships may represent the degree of communication weakness or transmission loss. In such a framework, N-fuzzy bridges correspond to critical communication links whose removal significantly affects network connectivity, whereas N-fuzzy cut nodes represent vulnerable devices whose failure may disrupt communication among other nodes. Therefore, the study of N-fuzzy bridges and cut nodes can help identify critical components of uncertain network structures.

The concept of an N-fuzzy tree plays a central role in the structural analysis of N-fuzzy graphs. Like classical graph theory, N-fuzzy trees provide a simplified framework for studying connectivity and path-related properties. However, due to the presence of negative membership values, several characteristics of N-fuzzy trees differ significantly from those of ordinary trees. The following section investigates these properties and presents various characterizations of N-fuzzy trees.

The results presented in this section demonstrate that several classical graph-theoretic notions can be extended to the N-fuzzy environment. However, the examples and counterexamples indicate that certain properties of N-fuzzy bridges and cut nodes differ from their classical counterparts. These observations emphasize the need for careful analysis when generalizing graph-theoretic results to N-fuzzy structures.

3. A Proposed Structure of N-Fuzzy Trees

A tree is a connected graph with no cycles. In a tree, every pair of points is connected by a unique path. Each vertex has at most one incoming edge. Rosenfeld has proved that if there exists a unique strongest path joining any two nodes in G, then G is a fuzzy tree, and the converse does not hold. Similarly, in N-fuzzy sets, if there exists a unique strongest path joining any two nodes in G, then G is an N-fuzzy tree, and the converse does not hold. In Fig. 3.1 (a), G is an N-fuzzy tree and P_1: x, u, v, w, y & P_2: x, v, w, and y are two strongest x–y paths with $\alpha^{\propto}$(u, v) =- .5 of which P_2 is in F Fig. 3.1 (b). Also note that if G^* is a tree, then F is G itself, and the maximum spanning tree T of an N-fuzzy tree G is the required N-fuzzy spanning subgraph, and T is unique for an N-fuzzy tree.

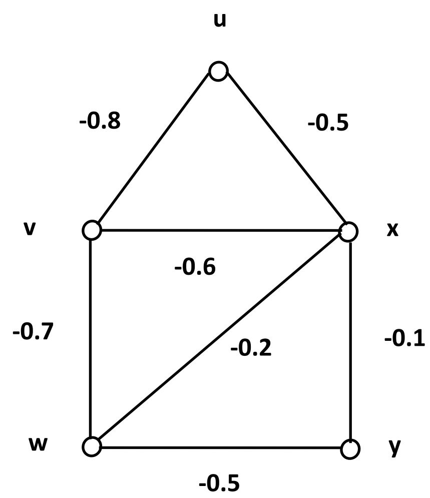

Fig.13. An N-fuzzy graph G

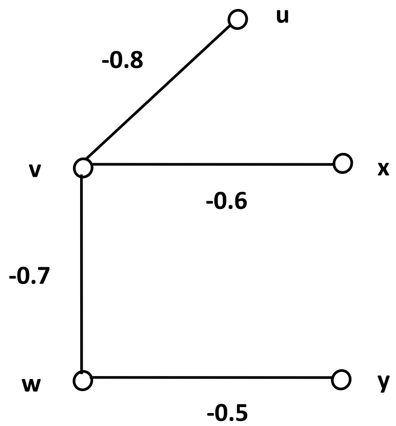

Fig.14. A Tree F in N-fuzzy graph G

Lemma 3.1 If (g, β) is a partial N-fuzzy subgraph of (f, α), then for all u, v $\beta^{\propto}(u, v) \geq \alpha^{\propto}(u, v)$.

Theorem 3.1 If G: (f, α) is an N-fuzzy tree and G* (f^*, α^*) is not a tree, then there exists at least one arc (u, v) in α^* for which $\alpha^{\infty}(u, v) < \alpha(u, v)$.

Proof: If G is an N-fuzzy tree, then there exists an N-fuzzy spanning subgraph F:(f, α) which is a tree and $\alpha^{\infty}(u, v) < \alpha(u, v)$ for all arcs (u, v) not in F. Also $\beta^{\propto}(u, v) \geq \alpha^{\propto}(u, v)$ by Lemma 3.3. Thus $\alpha^{\infty}(u, v) < \alpha(u, v)$ for all (u, v) not in F, and by hypothesis, there exists at least one arc (u, v) not in F, which completes the proof.

Theorem 3.2 Let G: be an N-fuzzy tree and G* $\neq K_1$. Then G is not complete.

Proof: If possible, let G be a complete N-fuzzy graph. Then $\alpha^{\infty}(u, v) = \alpha(u, v)$ for all u, v. [Lemma 3.1] Now, G being an N-fuzzy tree, $\alpha^{\infty}(u, v) < \alpha(u, v)$ for all (u, v) not in F. Thus $\alpha^{\propto}(u, v) > \beta^{\propto}(u, v)$, contradicting Lemma 3.3.

Remark 3.1 It is proved that if G is an N-fuzzy tree, then arcs of F are the N-fuzzy bridge of G, and thus F is unique. In the next theorem, we characterize an N-fuzzy tree.

Theorem 3.4 If G is an N-fuzzy tree, then internal nodes of F are the N-fuzzy cut nodes of G.

Proof: Let w be any node in G. which is not an end node of F. Then, it is the common node of at least two arcs in F which are the N-fuzzy bridges of G, and by Theorem 3.3, w is an N-fuzzy cut node. Also, if w is the ending node of F, then w is not an N-fuzzy cut node; for if so, and there exist u, v distinct from w such that w is on every strongest u-v path and one such path certainly lies in F. But w being an end node of F, this is not possible.

With reference to Remark 3.1, we have,

Corollary: An N-fuzzy cut node of an N-fuzzy tree is the common node of at least two N-fuzzy bridges.

We have seen that the condition that an arc (u, v) is the strongest u-v path is only necessary for it to be an N-fuzzy bridge [Theorem 3.1]. In the case of N-fuzzy trees, this condition also becomes sufficient, and we have the following characterization of N-fuzzy trees.

Theorem 3.5 G: $(f,\ \alpha)$ is an N-fuzzy tree if and only if the following are equivalent.

(1) Arc (u, v) is an N-fuzzy bridge.

(2) $\alpha^{\propto}$(u, v) = α (u, v).

Proof: Let G: $(f,\ \alpha)$ be an N-fuzzy tree, and let (u, v) be an N-fuzzy bridge. Then $\alpha^{\propto}$ (u, v) = α (u, v) [Theorem 3.1]. Now let (u, v) be an arc of G such that, $\alpha^{\propto}$(u, v) = α (u, v). If G* is a tree, then clearly (u, v) is an N-fuzzy bridge; otherwise, it follows from Theorem 3.8 that (u, v) is in F and it is an N-fuzzy bridge, [Theorem 3.4].

Conversely, assume that (1) ⟺ (2). Construct a maximum spanning tree T: $(f,\ \beta)$ for G. If (u, v) is in T, by an algorithm $\alpha^{\propto}$ (u, v) = α (u, v) and hence (u, v) is an N- fuzzy bridge. Now these are the only N-fuzzy bridges of G for, if possible, let (u′, v′) be an N-fuzzy bridge of G which is not in T. Consider a cycle C consisting of (u′, v′) and the unique u′-v′ path in T. Now, the arcs of this u′-v′ path being N-fuzzy bridges are not weakest arcs of C and hence (u′, v′) must be the weakest arc of C and hence cannot be an N-fuzzy bridge [theorem3.4].

Moreover, for all arcs (u′, v′) not in T, we have α (u′, v′) $>$ $\beta^{\propto}$(u′, v′) for, if possible, let but α (u′, v′) $\leq$ $\beta^{\propto}(u', v')$. But α (u′, v′) $>$ $\beta^{\propto}(u', v')$ (strict inequality holds since (u′, v′) is not an N-fuzzy bridge). So $\beta^{\propto}(u', v') > \alpha^{\propto}$ (u′, v′) which gives a contradiction, since $\beta^{\propto}(u',\ v')$ is the strength of the unique u′-v′ path in T, and by an algorithm $\beta^{\propto}$ (u′, v′) = $\alpha^{\propto}$(u′, v′) Thus, T is the required spanning subgraph F, which is a tree, and hence G is an N-fuzzy tree.

Remark 3.2 It follows from the proof of Theorem 3.5 that arcs of the maximum spanning T are the N-fuzzy bridges of the N-fuzzy bridges of the N-fuzzy tree G, and thus we have,

Theorem 3.6 An N-fuzzy graph is an N-fuzzy tree if and only if it has a unique maximum spanning tree.

Note that G is an N-fuzzy graph on n nodes, then the maximum number of N-fuzzy bridges in G is n-1 [Corollary to Theorem 3.4], and it follows from Remark 3.2 that an N-fuzzy tree on n nodes has (n-1) N-fuzzy bridges.

In the following theorem, we consider the case of a general N-fuzzy graph.

Theorem 3.6 Let G: $(f,\ \alpha)$ be a connected N-fuzzy graph with no N-fuzzy cycles. Then G is an N-fuzzy tree.

Proof: If G* has no N-fuzzy cycles, then G* is a tree, and G is an N-fuzzy tree. So, assume that G has cycles and by hypothesis no cycle is an N-fuzzy cycle, *i.e.,* every cycle in G will have exactly one weakest arc in it. Remove the weakest arc (say) e in a cycle C of G. If

There are still cycles in the resulting N-fuzzy graph, repeating the process, which will eventually result is an N-fuzzy subgraph, which is a tree, and which is the required spanning subgraph F, hence proved.

Remark 3.3 The converse of the above theorem is not true. In Fig.3.2, G is an N-fuzzy tree and u, v, w, u is an N-fuzzy cycle.

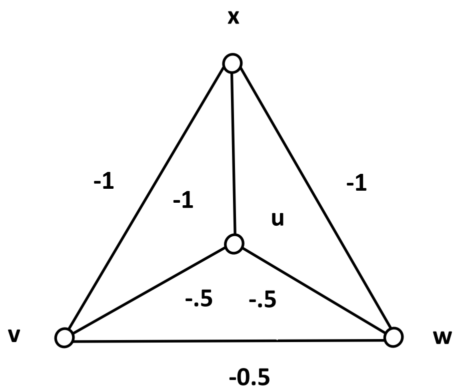

Fig.15. Graph G has an N-fuzzy cycle

4. Conclusions

In this paper, we investigated several structural aspects of N-fuzzy graphs. The notions of N-fuzzy bridges, N-fuzzy cut nodes, N-fuzzy cycles, and N-fuzzy trees were examined, and their interrelationships were discussed. Characterizations based on the strongest paths and the maximum spanning trees were presented. Furthermore, counterexamples were provided to demonstrate that certain converse statements involving N-fuzzy cut nodes and N-fuzzy trees do not hold in general.

The results obtained in this work contribute to a better understanding of the structural properties of N-fuzzy graphs and provide a foundation for further investigations. Future work may focus on developing N-fuzzy graph algorithms, connectivity measures, network reliability analysis, and applications of N-fuzzy graphs in communication networks, decision-making systems, and complex network analysis.

The results obtained in this work provide a basis for future studies on connectivity analysis, network vulnerability assessment, and structural optimization in N-fuzzy graph models. Further investigations may focus on algorithmic aspects, connectivity indices, and applications of N-fuzzy graphs in communication and decision-support systems.

4.1 Future Research Directions

The results presented in this work provide several opportunities for future investigation. One possible direction is the development of connectivity indices, vulnerability measures, and reliability models for N-fuzzy graphs. Another promising area is the study of shortest paths, spanning structures, and optimization problems in N-fuzzy environments. Furthermore, recent advances in intuitionistic fuzzy sets, hesitant fuzzy sets, and Pythagorean fuzzy models [19,22–24] suggest the possibility of constructing more generalized graph structures capable of handling higher levels of uncertainty. Such extensions may facilitate applications in decision-making, intelligent systems, medical diagnosis, supply chain optimization, communication networks, and artificial intelligence [17,20,21, 26]. Future research may also focus on algorithmic aspects of N-fuzzy graph theory and its integration with modern fuzzy decision-support frameworks. Moreover, it can adopted to evaluate several problems in different fields, such as, transport [27], energy [28. 29] and medical [30].

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